Chap 2. Perfect Conductivity

2.1 Introduction

- In this chapter, we begin modeling the superconductor using Maxwell equation.
- Superconductor will be considered to the simple R-L-C circuits.

CONTENTS

2.

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=0 \Leftrightarrow m\frac{d^2x}{dt^2}+R\frac{dx}{dt}+kx=0$$

To understand why time constants are a useful concept, we begin by examining the R-L-C circuit, as shown in Figure 2.1.

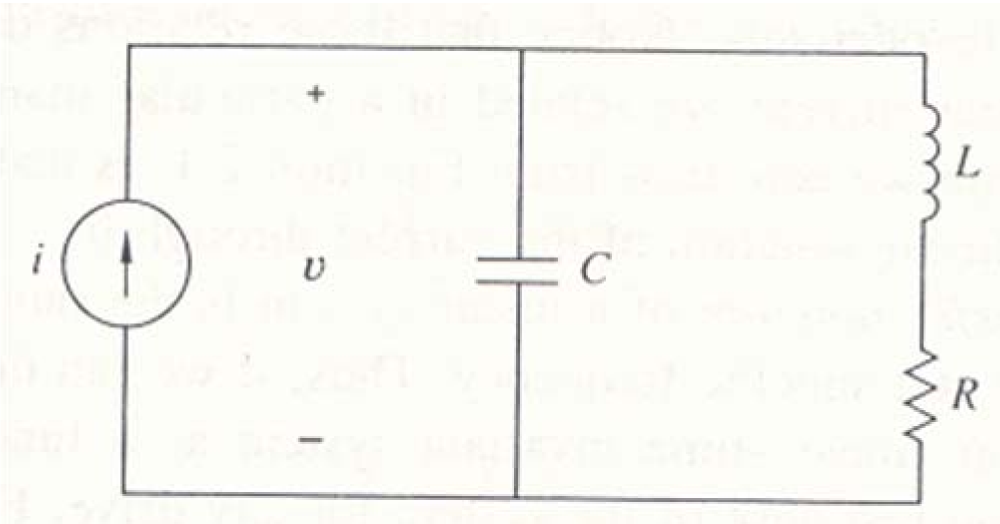


Figure 2.1

An R-L-C Circuit

We can express this voltage-current relation for;

the resistor as $v_R = i_R R$ ---------- (2.1)

the inductor as $v_L = L\frac{d}{dt}i_L$ ---------- (2.2)

the capacitor as $i_C = C\frac{d}{dt}v_C$ ---------- (2.2)

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=0 \Leftrightarrow m\frac{d^2x}{dt^2}+R\frac{dx}{dt}+kx=0$$

◈ We now find the impedance of our R-L-C circuit as seen by the source.

The drive of the system is an oscillation current that we can represent using complex numbers :

$$i=|i|\cos(\omega t+\phi)=\mathrm{Re}\{\hat{i}e^{j\omega t}\} \quad \text{— (2. 4)}$$

where

$$\hat{i} \equiv |i|e^{j\phi} \quad \text{— (2. 5)}$$

Because the system is linear and time invariant, this voltage will oscillate with the same frequency as the drive:

$$v=\mathrm{Re}\{\hat{v}e^{j\omega t}\} \quad \text{— (2. 6)}$$

We can therefore write

$$Z(\omega)=\frac{\hat{v}}{\hat{i}} \quad \text{— (2. 7)}$$

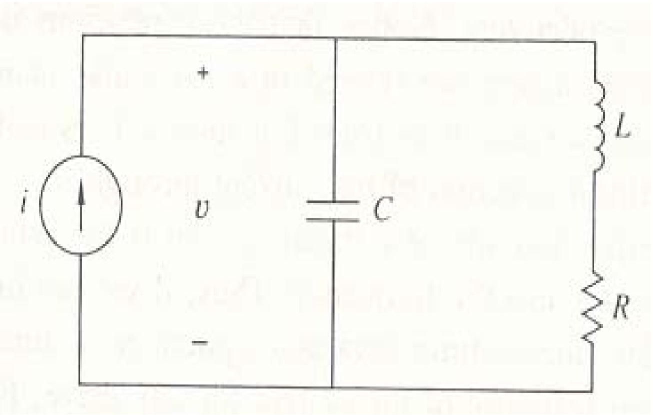
where Z (impedance) can be frequency dependent.

From $i = \mathrm{Re}\{\hat{i} e^{j\omega t}\}$ and $v = \mathrm{Re}\{\hat{v} e^{j\omega t}\}$

$$v_L = L \frac{di_L}{dt} = j\omega L i_L \quad \text{— (2. 8)}$$

$$i_C = C \frac{dv_C}{dt} = j\omega C\, v_C \quad \text{— (2. 9)}$$

We can solve for Z(ω) :



$$\frac{1}{Z(\omega)} = \frac{1}{R + j\omega L} + \frac{1}{1/j\omega C} = \frac{R + j\omega L + (\frac{1}{j\omega C})}{(R + j\omega L)(\frac{1}{j\omega C})}$$

$$Z(\omega) = \frac{(R + j\omega L)(\frac{1}{j\omega C})}{R + j\omega L + (\frac{1}{j\omega C})} = \frac{j\omega L + R}{(j\omega)^2 LC + j\omega RC + 1} \quad \text{— (2. 10)}$$

We will be better able to interpret the transfer function if we write it in terms of symbols that have more physical meaning.

Let us therefore define

the inductive time constant :

$$\tau_{RL} \equiv \frac{L}{R} \qquad \text{— (2. 11)}$$

the capacitive time constant :

$$\tau_{RC} \equiv RC \qquad \text{— (2. 12)}$$

the coupling time constant :

$$\tau_{LC} \equiv \sqrt{LC} = \sqrt{\tau_{RL}\tau_{RC}} \qquad \text{— (2. 13)}$$

τ_{LC} as the inverse of the resonant frequency of an L-C network. From these definitions, we can rewrite the transfer function in the more suggestive form:

$$Z(\omega) = \frac{j\omega L + R}{(j\omega)^2 LC + j\omega RC + 1} \Rightarrow Z(\omega) = R\left(\frac{1 + j\omega\tau_{RL}}{(1-(\omega\tau_{LC})^2) + j\omega\tau_{RC}}\right) \qquad \text{— (2. 14)}$$

$$Z(\omega) = R\left(\frac{1 + j\omega\tau_{RL}}{(1-(\omega\tau_{LC})^2) + j\omega\tau_{RC}}\right) \quad -(2.14)$$

Resonance Frequency (ω_0):

Z(ω) has maximum **value at** $\omega_0 = 1/\tau_{LC}$

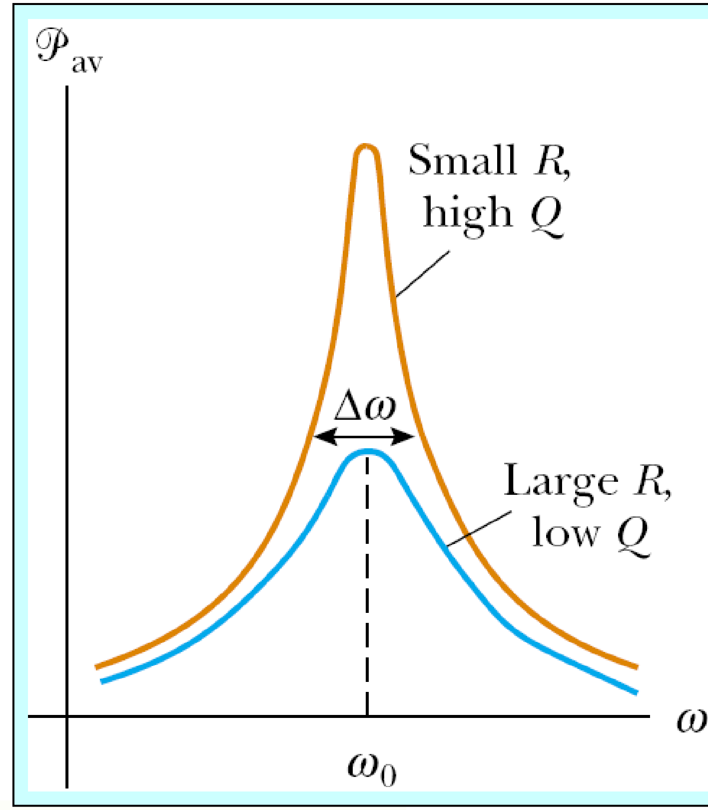
Low frequency limits :

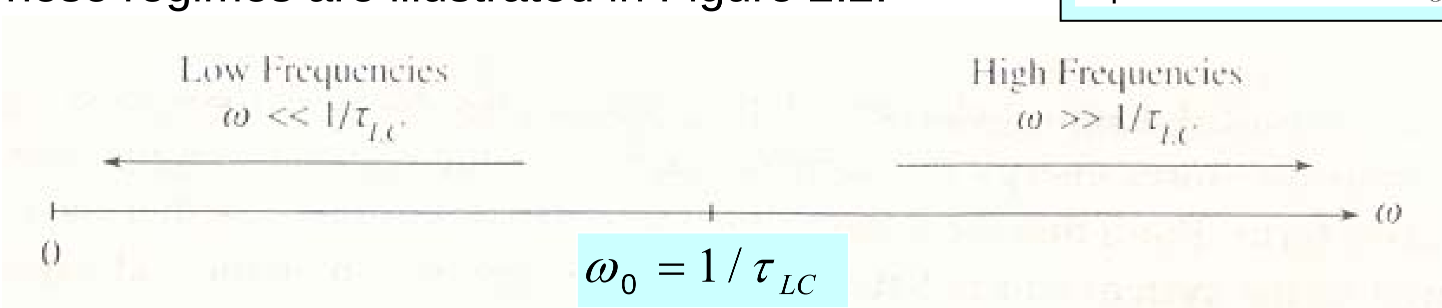
$$\omega << \omega_0 (= \frac{1}{\tau_{LC}}) = \quad \Rightarrow \quad \omega\tau_{LC} << 1 \quad ----(2.15)$$

High frequency limits :

$$\omega >> \omega_0 (= \frac{1}{\tau_{LC}}) = \quad \Rightarrow \quad \omega\tau_{LC} >> 1 \quad ----(2.16)$$

These regimes are illustrated in Figure 2.2.





$$Z(\omega) = R\left(\frac{1 + j\omega\tau_{RL}}{(1-(\omega\tau_{LC})^2) + j\omega\tau_{RC}}\right) - (2.14)$$

Approximation for low frequency limits (most cases)

$$\lim_{\omega\tau_{LC} << 1} Z(\omega) \approx R\left(\frac{1 + j\omega\tau_{RL}}{1 + j\omega\tau_{RC}}\right) \text{-----------------} (2.24)$$

1) For small resistance:

$$\omega\tau_{RC} << 1 \qquad \text{------------------} (2.25)$$

$$\lim_{\omega\tau_{RC} << 1} Z(\omega) \approx R(1 + j\omega\tau_{RL}) \text{-----------------} (2.26)$$

2) For large resistance:

$$\omega\tau_{RL} << 1 \qquad \text{------------------} (2.27)$$

$$\lim_{\omega\tau_{RL} << 1} Z(\omega) \approx R\left(\frac{1}{1 + j\omega\tau_{RC}}\right) \text{-----------------} (2.28)$$

2.3 Maxwell Equations

In section 2.2, we saw how it was much easier to find the response of a circuit to a particular source if we knew that only certain frequencies were used to drive the system.

The goal of this section is to extend this technique to distributed or macroscopic systems.

The expression describes the Maxwell's equation.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{— (2. 29)}$$

— Faraday's law

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad \text{— (2. 30)}$$

— Ampère's law

$$\nabla \circ \vec{D} = \rho \quad \text{— (2. 31)}$$

$$\nabla \circ \vec{B} = 0 \quad \text{— (2. 32)}$$

— Gauss's electric and magnetic law

Symbol	Name	Units
E	electric field	volts/meter
H	magnetic field	amps/meter
D	electric displacement	coulombs/(meter)^2
B	magnetic flux density	tesla
ρ	free charge density	coulombs/(meter)^3
J	free current density	amps/(meter)^2

The conversion is simple if we recall two mathematical theorems.

Stoke's theorem => $$\oint_C \mathrm{C} \cdot d\mathrm{l} = \int_S (\nabla \times \mathrm{C}) \cdot d\mathrm{s} \quad \text{— (2. 35)}$$

Gauss's theorem => $$\oint_S \mathrm{C} \cdot d\mathrm{s} = \int_V (\nabla \cdot \mathrm{C}) dv \quad \text{— (2. 36)}$$

Integral forms of Maxwell equation:

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t} \Rightarrow \oint_C \mathrm{E} \cdot d\mathrm{l} = -\frac{d}{dt}\int_S \mathrm{B} \cdot d\mathrm{s} \quad \text{— (2. 37)}$$

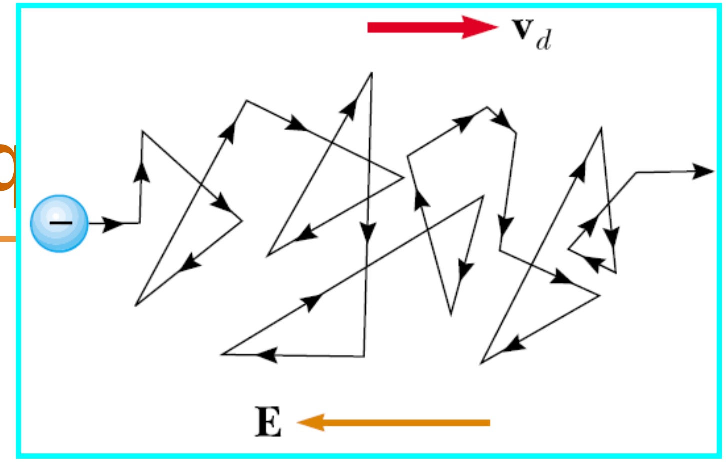
$$\nabla \times \mathrm{H} = \frac{\partial \mathrm{D}}{\partial t} + \mathrm{J} \Rightarrow \oint_C \mathrm{H} \cdot d\mathrm{l} = -\frac{d}{dt}\int_S \mathrm{D} \cdot d\mathrm{s} + \int_S \mathrm{J} \cdot d\mathrm{s} \quad \text{— (2. 38)}$$

$$\nabla \circ \mathrm{D} = \rho \Rightarrow \oint_S \mathrm{D} \cdot d\mathrm{s} = \int_V \rho dv \quad \text{— (2. 39)}$$

$$\nabla \circ \mathrm{B} = 0 \Rightarrow \oint_S \mathrm{B} \cdot d\mathrm{s} = 0 \quad \text{— (2. 40)}$$

$$\nabla \circ \mathrm{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \oint_S \mathrm{J} \cdot d\mathrm{s} + \frac{d}{dt}\int_V \rho dv = 0 \quad \text{— (2. 41)}$$

2.5 The first London eq



The goal of this section is to develop such a simple model of a true perfect conductor.

We start by examining a single carrier, an electron of mass m and velocity v, and consider its motion by EM force under Newtonian mechanics;

$$m\frac{d\mathrm{v}}{dt} = \mathrm{f}_{\mathrm{em}} + \mathrm{f}_{\mathrm{drag}} \qquad \text{— (2. 115)}$$

where the only forces assumed to be action on the particle are those resulting from an externally applied electromagnetic field, f_{em}, and those resulting from collision, f_{drag}.

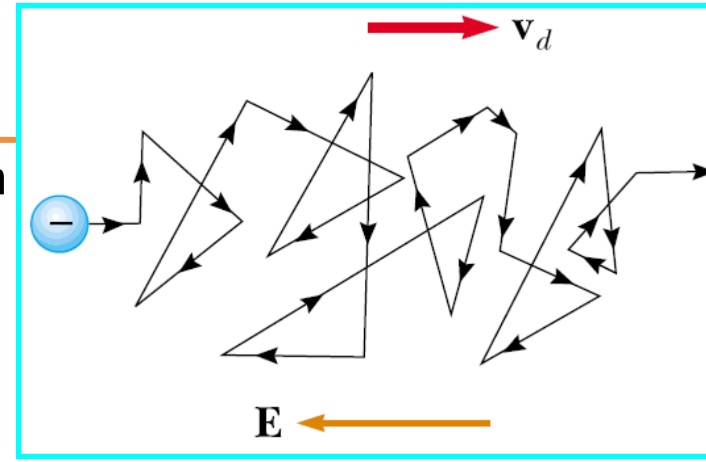
In general, for an object of charge q moving with a velocity v in an electromagnetic field, Lorentz's law states that the force on the object is

$$\mathrm{f}_{\mathrm{em}} = q(\mathrm{E} + (\mathrm{v} \times \mathrm{B})) \qquad \text{— (2. 116)}$$

The magnetic field will never contribute a force in

particle is moving.

$$f_{em} \approx qE$$



Drude proposed f_{drag}.

$$f_{drag} = -\frac{d\,p}{dt}$$

* τ_{tr} : scatter

With our forces thus defined,

expression for the velocity of

$$m\frac{d\mathrm{v}}{dt} = f_{em} + f_{drag} \Rightarrow \qquad m$$

$$\hat{v} = \left(\frac{q\tau_{tr}}{m}\right)\frac{1}{1 + j\omega\tau_{tr}}\hat{E}$$

$$m\frac{d\mathrm{v}}{dt} + \frac{m}{\tau_{tr}}\mathrm{v} = qE,\ E = \hat{E}e^{j\omega t},\ \ \mathrm{v} = \hat{v}e^{j\omega t}$$

$$\Rightarrow m\frac{d}{dt}(\hat{v}e^{j\omega t}) + \frac{m}{\tau_{tr}}(\hat{v}e^{j\omega t}) = q\hat{E}e^{j\omega t}$$

$$\Rightarrow j\omega m\hat{v} + \frac{m}{\tau_{tr}}\hat{v} = q\hat{E} \quad \Rightarrow \hat{v}m(j\omega + \frac{1}{\tau_{tr}}) = q\hat{E}$$

$$\Rightarrow \hat{v}(j\omega\tau_{tr} + 1) = \frac{q\tau_{tr}}{m}\hat{E},\ \therefore \hat{v} = \left(\frac{q\tau_{tr}}{m}\right)\frac{1}{1 + j\omega\tau_{tr}}\hat{E}$$

$$\text{Current density (I/A): } \mathrm{J} = nq\mathrm{v}$$ — (2. 121)

where n, the number density of carriers in the material.

Combining Equation 2.120 and 2.121 yields

$$\mathrm{J} = \left(\frac{nq^2\tau_{tr}}{m}\right)\frac{1}{1+j\omega\tau_{tr}}\mathrm{E}$$

From $\hat{v} = \left(\frac{q\tau_{tr}}{m}\right)\frac{1}{1+j\omega\tau_{tr}}\hat{E}$ ----(2.120)

$$= \sigma_0\left\{\frac{1}{1+j\omega\tau_{tr}}\right\}\mathrm{E}$$ — (2. 122)

where σ_0 defined as DC conductivity

$$\sigma_0(\omega=0) \equiv \frac{nq^2\tau_{tr}}{m} \propto n\tau_{tr}$$ — (2. 123)

Comparing this relation to the dispersive form of Ohm's law;
$(\mathrm{J}(\mathrm{r},\omega)=\sigma(\omega)\mathrm{E}(\mathrm{r},\omega))$, we obtain an expression for the AC conductivity:

$$\sigma(\omega) = \sigma_0\frac{1}{1+j\omega\tau_{tr}}$$ — (2. 124)

Because σ_0 as defined is an expression for the steady-state (dc) conductivity, a value that can be easily measured or found in tables, we can invert Equation 2.123 to get a useful expression for the scattering time:

$$\tau_{tr} = \frac{m\sigma_0}{nq^2} \qquad \text{from } \sigma_0 \equiv \frac{nq^2\tau_{tr}}{m} \qquad \text{— (2. 125)}$$

To obtain estimates of τ_{tr}, we choose copper.

We find the scattering time of copper,

$$\tau_{tr} \approx 2.4 \times 10^{-14} \text{ sec}$$

As a result, for frequencies as great as 1 THz, $\omega\tau_{tr} \ll 1$, and the conductivity of copper is independent of frequency.

$$\sigma(\omega) = \sigma_0 \frac{1}{1 + j\omega\tau_{tr}} \approx \sigma_0$$

We wish to find an expression similar to Ohm's law but in the limit $\tau_{tr} \to \infty$.

$$m\frac{d\text{v}}{dt} + \frac{m}{\tau_{tr}}\text{v} = q\text{E} \qquad \tau_{tr} \to \infty$$

$$m\frac{d\text{v}}{dt} = q\text{E} \quad (\text{v} = \frac{\text{J}}{nq} \Leftarrow \text{J} = nq\text{v})$$

$$\frac{m}{nq}\frac{d\text{J}}{dt} = q\text{E} \quad \Rightarrow \quad \text{E} = \frac{\partial}{\partial t}(\frac{m}{nq^2}\text{J})$$

$$\boxed{\Rightarrow \text{E} = \frac{\partial}{\partial t}(\Lambda \text{J})} \quad \Leftrightarrow \text{E} = \rho \text{J} \qquad \text{— (2. 131)}$$

where we define Λ as

$$\Lambda \equiv \frac{m^*}{n^* q^{*2}} \qquad \text{— (2. 132)}$$

This special form of Ohm's law was first proposed by Heinz and F. London in 1935 and is referred to as the "First London Equation".

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \Leftarrow \quad E = \frac{\partial}{\partial t}(\Lambda J)$$

$$\nabla \times E = \frac{\partial}{\partial t}\Lambda(\nabla \times J) = -\frac{\partial}{\partial t}\Lambda(\nabla^2 H)$$

$$-\frac{\partial}{\partial t}\Lambda(\nabla^2 H) = -\frac{\partial B}{\partial t} \quad (\leftarrow B = \mu_0 H)$$

$$\left(\frac{\mu_0}{\Lambda} - \nabla^2\right)\frac{\partial}{\partial t}H = 0$$

To properly solve a problem involving a must use the first London equation inst

Let us find the equation governing the a perfectly conducting system.

$$\nabla \times \nabla \times C = \nabla(\nabla \cdot C) - \nabla^2 C$$

$$\nabla \times \nabla \times H = \nabla(\nabla \cdot H) - \nabla^2 H = \nabla \times J$$

$$\nabla \times J = -\nabla^2 H \qquad \text{— (2. 133)}$$

The goal is to again express the current density in terms of the magnetic field, only time we use the first London equation.

$$\left(\frac{\mu_0}{\Lambda} - \nabla^2\right)\frac{\partial}{\partial t}H = 0 \qquad \text{— (2. 134)}$$

$\sqrt{\Lambda/\mu_0}$: a characteristic length associated with the properties of the system, .

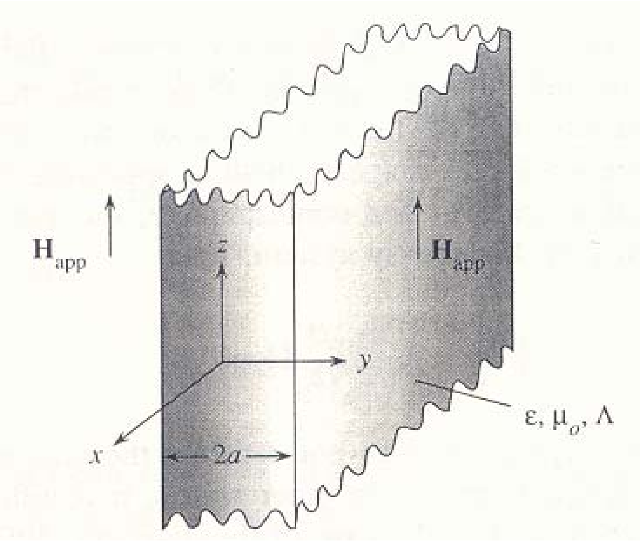
2.6 Field inside a perfect conductor

We look at a homogeneous, isotropic, linear perfectly conducting slab (ε, μ_0, Λ) that is subjected on both sides to the applied magnetic field

$$\mathrm{H}_{\mathrm{app}} = H_0 \cos(\omega t)\mathrm{i}_z = \mathrm{Re}\{H_0 e^{j\omega t}\}\mathrm{i}_z \quad \text{— (2. 135)}$$

as shown in Figure 2.24.

As in that section, our goal is to find the magnetic field and current density inside the slab.



☜Figure 2.24 An infinite slab of finite thickness. The slab is perfect conductor.

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

We know the time dependence of the driving field, H_{app} is sinusoidal and so can express Equation 2. 134 as

$$\left(\frac{\mu_0}{\Lambda} - \nabla^2\right)\frac{\partial}{\partial t}\mathrm{H} = 0, \qquad \mathrm{H}_{app} = \mathrm{Re}\{H_0 e^{j\omega t}\}\mathrm{i}_z$$

$$\Rightarrow j\omega\left(\frac{\mu_0}{\Lambda} - \frac{d^2}{dy^2}\right)\mathrm{H}(y) = 0$$ for an infinite slab — (2. 136)

For nonzero frequencies ($\omega \neq 0$), this relation simplifies to

$$\left(\frac{\mu_0}{\Lambda} - \frac{d^2}{dy^2}\right)\mathrm{H}(y) = 0 \Rightarrow \left(\frac{1}{\lambda^2} - \frac{d^2}{dy^2}\right)\mathrm{H}(y) = 0$$ — (2. 137)

Solution is given by $\mathrm{H}(y) = C\cosh(y/\lambda)$ — (2. 138)

where we defined the penetration depth (skin depth δ of magnetic diffusion)

$$\lambda \equiv \sqrt{\frac{\Lambda}{\mu_0}}$$ — (2. 141)

Solution: $H(y) = C\cosh(y/\lambda)$

Such as Nb-Ti and Nb_3Sn, as well as the high-T_c superconductors, λ is typically on the order of 100 nm.

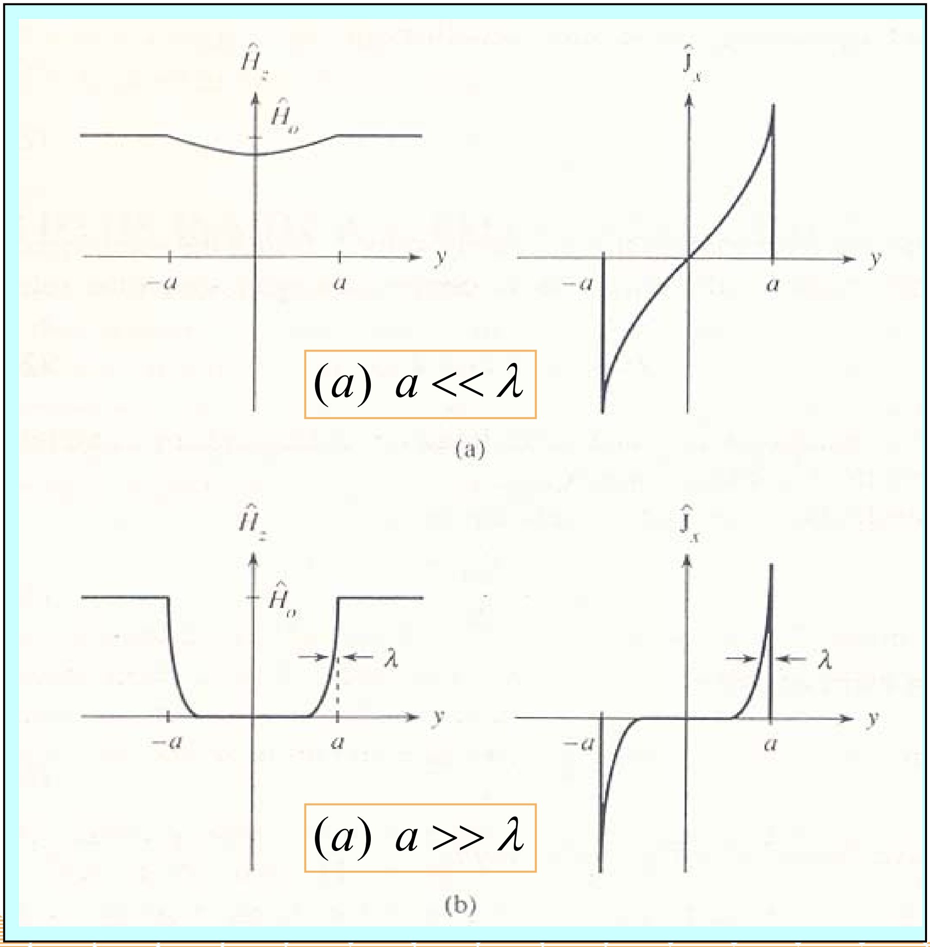
Boundary condition at $y = (+a, -a)$ gives

$$C = H_0(y = a)\frac{1}{\cosh(a/\lambda)} \quad \text{— (2. 142)}$$

$$H = \mathrm{Re}\{H_0 \frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} e^{j\omega t}\}\mathrm{i}_z \quad \text{— (2. 143)}$$

$$J = \nabla \times H \Rightarrow$$

$$J = \mathrm{Re}\{H_0 \frac{\sinh(y/\lambda)}{\cosh(a/\lambda)} e^{j\omega t}\}\mathrm{i}_x \quad \text{— (2. 144)}$$



☜ Figure 2.25 The distribution of field and current density in a perfect conductor; (a) (a/λ) << 1 and (b) (a/λ) >> 1.

Calculations of Current density

$$\mathrm{J} = \nabla \times \mathrm{H},\ \ \mathrm{H} = \mathrm{Re}\{H_0 \frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} e^{j\omega t}\}\mathrm{i}_z$$

$$\nabla \times \mathrm{H} = \begin{pmatrix} \mathrm{i}_x, & \mathrm{i}_y, & \mathrm{i}_z \\ \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z} \\ 0, & 0, & H_0 \frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} e^{j\omega t} \end{pmatrix}$$

$$= \mathrm{i}_x \left(\frac{\partial}{\partial y} \left\{ H_0 \frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} e^{j\omega t} \right\} - 0 \right) + \mathrm{i}_y \left(0 - \frac{\partial}{\partial x} \left\{ H_0 \frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} e^{j\omega t} \right\} \right) + \mathrm{i}_z (0 - 0)$$

$$= \mathrm{Re}\{H_0 \frac{\sinh(y/\lambda)}{\cosh(a/\lambda)} e^{j\omega t}\}\mathrm{i}_x$$

For DC magnetic field:

$$\mathrm{H}_{\mathrm{app}} = H_0 \mathbf{i}_z \qquad \text{— (2. 145)}$$

Let us carry this integration out for the specific case of our infinite slab subjected to a tangential magnetic field. From Equation 2.134, we obtain the differential equation:

$$\left(\frac{\mu_0}{\Lambda} - \nabla^2\right)\frac{\partial}{\partial t}\mathrm{H} = 0$$

$$\Rightarrow \left(\frac{\mu_0}{\Lambda} - \frac{\partial^2}{\partial y^2}\right)\frac{\partial}{\partial t}\mathrm{H}(y,t) = 0 \qquad \text{— (2. 146)}$$

Again we find the spatial variation

$$\frac{\partial}{\partial t}H(y,t) = C(t)\cosh(y/\lambda) \qquad \text{— (2. 147)}$$

$$\frac{\partial}{\partial t} H(y,t) = C(t)\cosh(y/\lambda) \quad \text{— (2. 147)}$$

where C(t) is constant in space, but it is time dependent.

At the boundary planes y = (+a, -a); $C(t) = \frac{\partial}{\partial t} H(a,t)/\cosh(a/\lambda)$

$$\frac{\partial}{\partial t} H(y,t) = \frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} \frac{\partial}{\partial t} H(a,t) \quad \text{— (2. 148)}$$

The general solution to our perfectly conducting slab problem is therefore

$$\Delta H(y,t) = \frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} \Delta H(a,t)$$

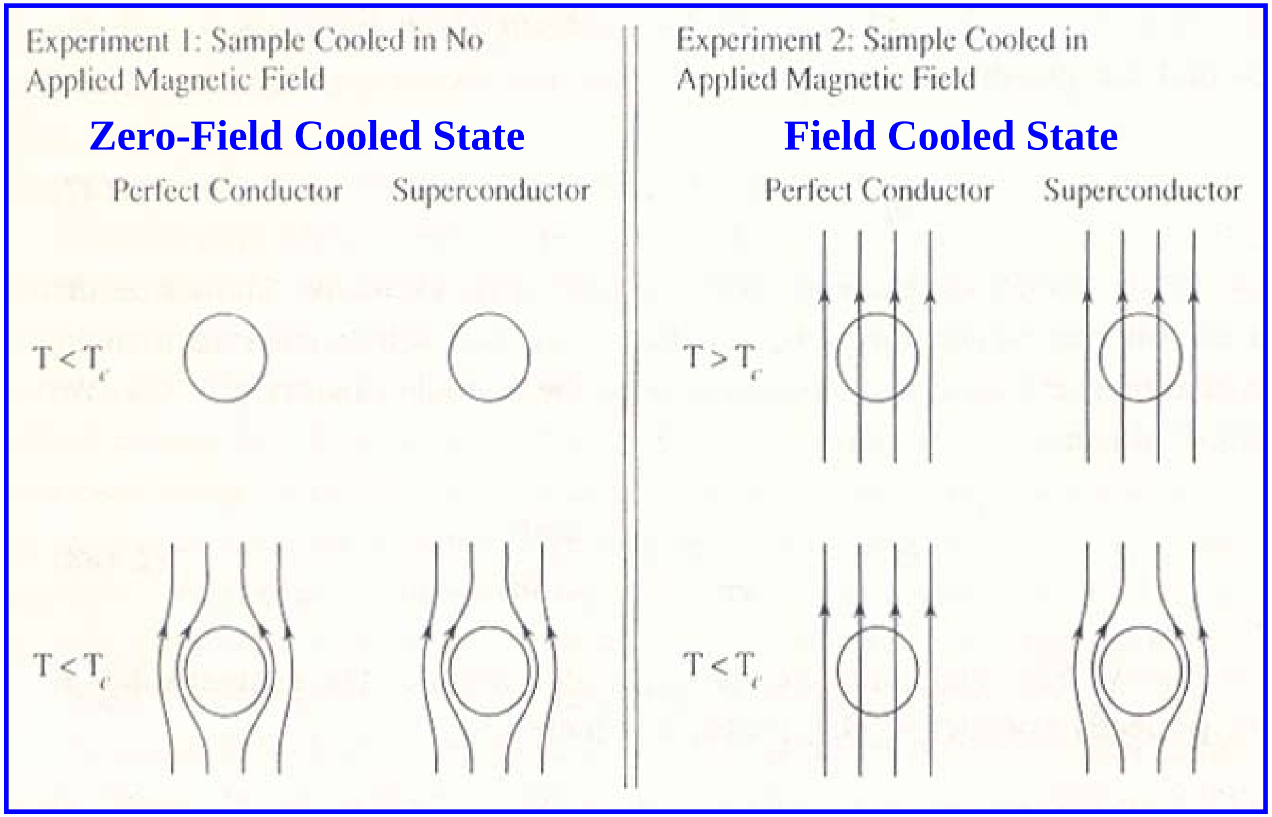
$$H(y,t) - H(y,0) = [H(a,t) - H(a,0)] \frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} \quad \text{— (2. 149)}$$

Perfect conductor vs Superconductor

$$H(y,t) = [H(a,t) - H(a,0)]\frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} + H(y,0),$$

$$H(a,t) = H(a,0) \text{ for DC field} \Rightarrow H(y,t) = H(y,0)$$

"Field in a sample is always independent of time"



A perfect conductor is a flux conserving medium.

A superconductor is a flux expelling medium.