Chap. 5 Macroscopic Quantum Model

Outline

1. Introduction
2. Schrodinger's equation
3. Probability currents
4. Macroscopic Quantum currents
5. Flux Quanta
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5-1 Introduction

-Superconductivity is a *Macroscopic Quantum Phenomenon*: there is no way to account for the phenomenon under Classical Physics only.

☞ The central hypothesis behind the MQM:
There exists a *Macroscopic Quantum Wave function, $\Psi(r,t)$,* that describes the behavior of the entire ensemble of superelectrons in superconductor.

The chapter is organized as follows:

- Introduce the most basic notion of Quantum Physics (wave-particle duality of nature).
- How a single quantum particle moves?.
- The net motion of ensemble gives us supercurrent (SC).
- Superconducting behavior.

5-2 Schrodinger's equation – Free particle

Wave-like properties

$$\varepsilon = \hbar\omega \qquad \mathrm{p} = \hbar\mathrm{k}$$

frequency | Planck's const. | Wave vector

Particle-like properties

$$\varepsilon = \tfrac{1}{2} m(\mathrm{v}\cdot\mathrm{v}) = \tfrac{1}{2m}\mathrm{p}\cdot\mathrm{p} \qquad \mathrm{p} = m\cdot\mathrm{v}$$

energy | momentum

Combine wave & particle properties we get the *dispersion relation* for the matter waves:

$$\varepsilon = \hbar\omega = \tfrac{\hbar^2}{2m}(\mathrm{k}\cdot\mathrm{k})$$

Assuming that the uniform *plane wave* satisfy the dispersion relation

$$\psi = \hat{\psi}\, e^{i(\mathrm{k}\cdot\mathrm{r}-\omega t)}$$

The differential equation (Schrodinger Eq.) that gives the dispersion relation is

$$i\hbar\tfrac{\partial}{\partial t}\psi = -\tfrac{\hbar^2}{2m}\nabla^2\psi \quad \text{for free partices}$$

This is justified by experimental confirmation not by derivation.

5-2 Schrodinger's equation – with forces

We have present a plausibility argument (not derivation) to show the connection between the classical and quantum formulation.

The energy of particle is described by: $\varepsilon = \frac{1}{2}m(\mathrm{v}\cdot\mathrm{v}) + V(\mathrm{r})$

Total energy of particle is conserved if the potential is <u>independent of time</u>

$$0 = \frac{d\mathcal{E}}{dt} = m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} + \frac{d}{dt}V(\mathbf{r})$$

$$= m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} + \underbrace{\cancel{\frac{\partial}{\partial t}V(\mathbf{r})}}_{0} + \underbrace{(\mathbf{v}\cdot\nabla)V(\mathbf{r})}_{\text{Chain rule}}$$

$$= \mathbf{v}\cdot\left(m\frac{d\mathbf{v}}{dt} + \nabla V\right) \Rightarrow \boxed{m\frac{d\mathbf{v}}{dt} = -\nabla V}$$

This is simply Newton's law of motion

5-2 Schrodinger's equation & Canonical Momentum

$$\frac{d\mathbf{p}}{dt} = -\nabla V$$

$$\frac{d}{dt}\Big(\underbrace{\text{Canonical} - \text{momentum}}_{\mathrm{p} = m\mathrm{v}}\Big) = -\nabla\Big(\underbrace{\text{Generalized} - \text{potential}}_{V}\Big)$$

In this case, the canonical momentum is identically equal to the kinetic momentum and the spatial derivative is the generalized potential. The energy of system is:

$$\mathcal{E} = \frac{1}{2m}(\mathbf{p}\cdot\mathbf{p}) + V(\mathbf{r})$$

We assumed that this classical expression is valid under QM. Therefore, it becomes

$$\hbar\omega = \frac{\hbar^2}{2m}(\mathbf{k}\cdot\mathbf{k}) + V(\mathbf{r}) \Rightarrow \underbrace{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi}_{\boxed{\text{Schrodinger's equation}}}$$

From Classical to Quantum expression

(1) Write the classical equation of motion (EOM) in terms of the canonical momentum **p** and generalized potential ***V***:

$$\frac{d\mathbf{p}}{dt} = -\nabla V$$

Indeed, this form identifies the precise expressions for **p** and ***V*****.**

(2) Use these quantities to write the energy of the system:

$$\mathcal{E} = \frac{1}{2m}(\mathbf{p}\cdot\mathbf{p}) + V$$

(3) Transform the classical expression into a QM one by appealing to the Einstein – de Broglie relations. Since Schrodinger's equation (S-Eq.) is linear, these transformations are:

$$\mathcal{E} = \hbar\omega \Longrightarrow i\hbar\frac{\partial}{\partial t} \quad \text{and} \quad \mathrm{p} = \hbar\mathrm{k} \Longrightarrow -i\hbar\nabla$$

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi} \quad \Leftarrow \psi = \hat{\psi}\, e^{i(\mathrm{k}\cdot\mathrm{r}-\omega t)}$$

5-3 Probability Currents

★ ***What is the properties of wave function ψ: Real or Complex?***.

Let's compare the plane wave solutions for *QM* and *Electromagnetism* cases:

$$\psi = \widehat{\psi}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi$$

The quantum wave function ψ must be a complex quantity.

$$\mathbf{H} = \mathrm{Re}\left\{\widehat{\mathbf{H}}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right\}$$

$$\nabla^2\mathbf{H} - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{H} = 0$$

In contrast, the electromagnetic fields H may be presented as either the real or imaginary parts of a complex expression.

5-3 Probability Currents (continued...)

★ ***What is the physical meaning of the quantum wave function ψ?:***

- The absolute of a plane wave should *not* influence the overall physics of a system. Max Born hypothesized in 1927 that the square of the magnitude of the wave function ψ was equal to the *probability* of a QM particle to be at the location **r** at time *t*.

$$\wp(\mathbf{r},t) \equiv |\psi(\mathbf{r},t)|^2 = \psi^*(\mathbf{r},t)\psi(\mathbf{r},t)$$

The probability of the location of the particle in space at a certain time

Since, the particle must exist *somewhere* in space; so, the wave function ψ must be satisfy the *normalization condition*:

$$\int d\mathbf{r}\, \psi^*(\mathbf{r},t)\psi(\mathbf{r},t) = 1$$

Probability Currents– Evolution of Probability

Let multiply the S-Eq by complex conjugate of ψ^* then subtract complex conjugate S-Eq* multiplied by ψ:

$$i\hbar\psi^* \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V\psi^*\psi$$

$$(-) \quad -i\hbar\psi \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\psi\nabla^2\psi^* + V\psi\psi^*$$

$$i\hbar \frac{\partial}{\partial t}(\psi\psi^*) = -\frac{\hbar^2}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right)$$

$\boxed{\text{By using vector product}: \nabla\cdot(\gamma \mathrm{C}) = \gamma\nabla\cdot\mathrm{C} + \mathrm{C}\cdot\nabla\gamma \quad \text{see next page...}}$

$$i\hbar\frac{\partial}{\partial t}\underbrace{(\psi\psi^*)}_{\mathcal{P}} = \underbrace{-\frac{\hbar^2}{2m}\left(\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\right)}_{\nabla\cdot\mathbf{J}_{\mathcal{P}}}$$

The left side of this expression as *time evolution of the probability* $\wp(\mathbf{r},t)$.

$\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) = \nabla\cdot\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)$ 유도하기

By using vector product rule :

$$\mathrm{C}\cdot\nabla\gamma = \nabla\cdot(\gamma\mathrm{C}) - \gamma\nabla\cdot\mathrm{C} \Rightarrow \gamma\nabla\cdot\mathrm{C} = \nabla\cdot(\gamma\mathrm{C}) - \mathrm{C}\cdot\nabla\gamma$$

Let's set; $\gamma = \Psi,\ \mathrm{C} = \nabla\Psi$

$$\Psi^*(\nabla\cdot\nabla\Psi) = \nabla\cdot(\Psi^*\nabla\Psi) - \nabla\Psi\cdot\nabla\Psi^*$$

$$\Psi(\nabla\cdot\nabla\Psi^*) = \nabla\cdot(\Psi\nabla\Psi^*) - \nabla\Psi^*\cdot\nabla\Psi$$

$$\begin{aligned}\left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) &= \Psi^*(\nabla\cdot\nabla\Psi) - \Psi(\nabla\cdot\nabla\Psi^*)\\ &= \nabla\cdot(\Psi^*\nabla\Psi) - \nabla\cdot(\Psi\nabla\Psi^*)\end{aligned}$$

$$\therefore \left(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*\right) = \nabla\cdot\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)$$

Probability Currents (cont.)

We find that the probability $\wp(\mathbf{r},t) \equiv |\psi(\mathbf{r},t)|^2 = \psi^*(\mathbf{r},t)\psi(\mathbf{r},t)$

and the probability current $\mathbf{J}_\wp \equiv \frac{\hbar}{2im}(\psi^*\nabla\psi - \psi\nabla\psi^*) = \mathrm{Re}\left\{\frac{\hbar}{im}\Psi^*\nabla\Psi\right\}$

$$\psi = \hat{\psi}\, e^{i(\mathrm{k}\cdot\mathrm{r}-\omega t)}$$

satisfy a continuing relation $\frac{\partial\wp}{\partial t} = -\nabla\cdot\mathbf{J}_\wp$

This is our desired relation describing the evolution of the probability of a quantum particle being found at a <u>certain point</u> space at a certain time.

In case of charged particles, we want that the classical Eq. of motion has general form

$$\frac{d}{dt}(\text{canonical momentum}) = -\nabla(\text{externally applied potential})$$

We start from the Lorentz's law for a particle of charge q in an EM field:

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + (\mathbf{v}\times\mathbf{B})\right)$$

using the vector potential $\mathbf{B} = \nabla\times\mathbf{A}$ & scalar potential $\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi$

to find $$\frac{d\mathbf{p}}{dt} = -\nabla\left(q\phi - \frac{q}{m}\mathbf{p}\cdot\mathbf{A} + \frac{q^2}{2m}\mathbf{A}\cdot\mathbf{A}\right)$$

☻ see next page

where canonical momentum is $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$

and the generalized potential: $\boldsymbol{V} = q\phi - \frac{q}{m}\mathbf{p}\cdot\mathbf{A} + \frac{q^2}{2m}\mathbf{A}\cdot\mathbf{A}$

The total energy of the system: $$\mathcal{E} = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + \left(q\phi - \frac{q}{m}\mathbf{p}\cdot\mathbf{A} + \frac{q^2}{2m}\mathbf{A}\cdot\mathbf{A}\right)$$

$$\boxed{\nabla\times\mathrm{E} = -\frac{\partial\mathrm{B}}{\partial t} \;\Rightarrow\; \nabla\times(\mathrm{E}+\frac{\partial\mathrm{A}}{\partial t}) = 0,\;\; \mathrm{B}=\nabla\times\mathrm{A}}$$

$$m\frac{d\mathrm{v}}{dt} = q\{(\mathrm{E}+(\mathrm{v}\times\mathrm{B})\},\;\; \text{Lorentz Eq.}$$

$$m\frac{d\mathrm{v}}{dt} = -q\left\{\nabla\phi+\frac{\partial\mathrm{A}}{\partial t}-\mathrm{v}\times(\nabla\times\mathrm{A})\right\} \Leftarrow \mathrm{E} = -\frac{\partial\mathrm{A}}{\partial t}-\nabla\phi$$

$$\text{chain rule}: \frac{\partial\mathrm{A}}{\partial t} = \frac{d\mathrm{A}}{dt}-(\mathrm{v}\cdot\nabla)\mathrm{A}$$

$$\frac{d}{dt}(m\mathrm{v}+q\mathrm{A}) = -q\{\nabla\phi-(\mathrm{v}\cdot\nabla)\mathrm{A}-\mathrm{v}\times(\nabla\times\mathrm{A})\}$$

$$\text{Canonical momentum}: \mathrm{p}\equiv m\mathrm{v}+q\mathrm{A}$$

By using the vector identities:

$$\mathrm{C}\times(\nabla\times\mathrm{D}) = \nabla(\mathrm{C}\cdot\mathrm{D})-(\mathrm{C}\cdot\nabla)\mathrm{D}-(\mathrm{D}\cdot\nabla)\mathrm{C}-\mathrm{D}\times(\nabla\times\mathrm{C})$$

$$\mathrm{v}\times(\nabla\times\mathrm{A}) = \nabla(\mathrm{v}\cdot\mathrm{A})-(\mathrm{v}\cdot\nabla)\mathrm{A}-(\mathrm{A}\cdot\nabla)\mathrm{v}-\mathrm{A}\times(\nabla\times\mathrm{v})$$

$$\boxed{\Rightarrow \frac{d\mathrm{p}}{dt} = -\nabla\left(q\phi-\frac{q}{m}(\mathrm{p}\cdot\mathrm{A})+\frac{q^2}{m}(\mathrm{A}\cdot\mathrm{A})\right)}$$

From $\mathcal{E} = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} + \left(q\phi - \frac{q}{m}\mathbf{p}\cdot\mathbf{A} + \frac{q^2}{2m}\mathbf{A}\cdot\mathbf{A}\right)$

Energy can be rewritten as $\mathcal{E} = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})\cdot(\mathbf{p} - q\mathbf{A}) + q\phi$

This expression is classical in origin. Now, we translate this classical expression to quantum one by using transformation as done before

$$\boxed{\mathcal{E} \Longrightarrow i\hbar\frac{\partial}{\partial t}} \quad \text{and} \quad \boxed{\mathbf{p} \Longrightarrow -i\hbar\nabla} \qquad = \mathrm{Re}\left\{\psi^* \frac{\hbar}{im}\nabla\psi\right\}$$

The Schrodinger's equation becomes

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2\psi + q\phi\,\psi} \qquad \Rightarrow \mathrm{p} = \frac{\hbar}{i}\nabla - q\mathrm{A}$$

Using S-Eq. to find the probability current of charged quantum particle in an electromagnetic field

$$\boxed{\mathbf{J}_{\wp} = \mathrm{Re}\left\{\psi^*\left(\frac{\hbar}{im}\nabla - \frac{q}{m}\mathbf{A}\right)\psi\right\}} \quad \Leftarrow \mathrm{J}_P = \mathrm{Re}\left\{\Psi^*\frac{\hbar}{im}\nabla\Psi\right\} = \mathrm{Re}\left\{\Psi^*\frac{1}{m}\mathrm{p}\Psi\right\}$$

5-4 Macroscopic Quantum Currents

❖ Assuming that the wavefunction describes the entire ensemble of superelectrons (N^*), it satisfies the relations

$$\int d\mathbf{r}\,\Psi^*(\mathbf{r},t)\Psi(\mathbf{r},t) = \mathsf{N}^\star \qquad \& \qquad \Psi^*(\mathbf{r},t)\Psi(\mathbf{r},t) = n^\star(\mathbf{r},t)$$

Total number of superelectrons

Local density of superelectrons

❖ With the *local constraint*, the flow of probability becomes the flow of particles. The macroscopic quantum current is given:

$$\mathbf{J}_\mathsf{S} = q^\star\,\mathrm{Re}\left\{\Psi^*\left(\frac{\hbar}{im^\star}\nabla - \frac{q^\star}{m^\star}\mathbf{A}\right)\Psi\right\}$$

❖ The macroscopic quantum wavefunction obeys a Schrodingerlike Eq.

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \frac{1}{2m^\star}\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}(\mathbf{r},t)\right)^2\Psi(\mathbf{r},t) + q^\star\phi(\mathbf{r},t)\,\Psi(\mathbf{r},t)$$

Writing the wavefunction in form of $\Psi(\mathbf{r},t) = \sqrt{n^\star(\mathbf{r},t)}\,e^{i\theta(\mathbf{r},t)}$ we find the *super-current equation*

$$\boxed{\mathbf{J}_\mathsf{S} = q^\star n^\star(\mathbf{r},t)\left(\frac{\hbar}{m^\star}\nabla\theta(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}(\mathbf{r},t)\right)}$$

The supercurrent equation (n* = constant)

Let $n^*(\mathrm{r},t) = n^*$ be a constant in space and time. The new macroscopic wavefunction has the form

$$\Psi(\mathbf{r},t) = \sqrt{n^\star}\, e^{i\theta(\mathbf{r},t)}$$

$$\boxed{\mathbf{J}_\mathrm{S} = q^\star n^\star(\mathbf{r},t)\left(\frac{\hbar}{m^\star}\nabla\theta(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}(\mathbf{r},t)\right)}$$

By using $\Lambda \equiv \dfrac{m^\star}{n^\star(q^\star)^2}$ then, $\Lambda\mathbf{J}_\mathrm{S} = -\left(\mathbf{A}(\mathbf{r},t) - \dfrac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right)$

Since n^* = constant in space & time, the Schrodinger-like equation is given by

$$\boxed{-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n^\star}\Lambda\,\mathbf{J}_\mathrm{S}{}^2 + q^\star\phi}$$

Energy-phase relationship

from
$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \frac{1}{2m^\star}\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}(\mathbf{r},t)\right)^2\Psi(\mathbf{r},t) + q^\star\phi(\mathbf{r},t)\,\Psi(\mathbf{r},t)}$$

London's equations

Both the 1st and 2nd London's equations are *direct consequences* of the super-current equation in the limit of constant n^*.

✓ By taking the curl of the supercurrent equation $\Lambda \mathbf{J_S} = -\left(\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right)$;

we can obtain the 2nd London's eqn: $\boxed{\nabla \times (\Lambda \mathbf{J_S}) = -\nabla \times \mathbf{A} = -\mathbf{B}}$

✓ We take the time derivative of the supercurrent equation:

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J_S}) = -\left[\frac{\partial \mathbf{A}}{\partial t} - \frac{\hbar}{q^\star}\nabla\left(\frac{\partial \theta}{\partial t}\right)\right]$$

By using the *energy-phase relationship*: $\boxed{-\hbar\frac{\partial \theta}{\partial t} = \frac{1}{2n^\star}\Lambda \mathbf{J_S}^2 + q^\star \phi}$

$$\boxed{\frac{\partial}{\partial t}(\Lambda \mathbf{J_S}) = \mathbf{E} - \frac{1}{n^\star q^\star}\nabla\left(\frac{1}{2}\Lambda \mathbf{J_S}^2\right)} \quad \text{where} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$$

The same as the 1st London's equation if E is dominant over the second term.

London's equations (continued...)

Lorentz force from first and second London equations

$$\frac{\partial}{\partial t}(\Lambda \mathbf{J}_S) = \mathbf{E} - \frac{1}{n^\star q^\star}\nabla\left(\frac{1}{2}\Lambda \mathbf{J}_S{}^2\right) \quad \text{and} \quad \nabla \times (\Lambda \mathbf{J}_S) = -\nabla \times \mathbf{A} = -\mathbf{B}$$

By using vector identities and $J_s = n^* q^* v^*_s$ and second London equation, in 1st London equation we get full Lorentz force (see next page);

$$m^\star \frac{d\mathbf{v}_S}{dt} = q^\star \mathbf{E} + q^\star \mathbf{v}_S \times \mathbf{B}$$

☞ This indicate that **the Meissner effect is a bit more fundamental in the physics of superconductivity than the phenomenon of zero resistance.**

$$\frac{\partial}{\partial t}(\Lambda \mathrm{J_s}) = \mathrm{E} - \frac{1}{n^* q^*}\nabla(\frac{1}{2}\Lambda \mathrm{J_s}^2), \quad \mathrm{C}\times(\nabla\times\mathrm{C}) = (1/2)\nabla(\mathrm{C}\cdot\mathrm{C}) - (\mathrm{C}\cdot\nabla)\mathrm{C}$$

$$\frac{\partial}{\partial t}(\Lambda \mathrm{J_s}) = \mathrm{E} - \frac{1}{n^* q^*}(\frac{1}{2}\Lambda)\nabla(\mathrm{J_s}\cdot\mathrm{J_s})), \; \nabla(\mathrm{J_s}\cdot\mathrm{J_s}) = 2\left\{\mathrm{J_s}\times(\nabla\times\mathrm{J_s}) + (\mathrm{J_s}\cdot\nabla)\mathrm{J_s}\right\}$$

$$\frac{\partial}{\partial t}(\Lambda \mathrm{J_s}) = \mathrm{E} - \frac{1}{n^* q^*}(\frac{1}{2}\Lambda)2\left\{\mathrm{J_s}\times(\nabla\times\mathrm{J_s}) + (\mathrm{J_s}\cdot\nabla)\mathrm{J_s}\right\} \Leftarrow \nabla\times\mathrm{J_s} = -\mathrm{B}/\Lambda$$

$$\frac{\partial}{\partial t}(\Lambda \mathrm{J_s}) + (\mathrm{v_s}\cdot\nabla)(\Lambda\mathrm{J_s}) = \mathrm{E} + \frac{1}{n^* q^*}(\mathrm{J_s}\times\mathrm{B}) \quad \Leftarrow \mathrm{J_s} = n^* q^* \mathrm{v_s}$$

$$\frac{d}{dt}(\Lambda \mathrm{J_s}) = \frac{\partial}{\partial t}(\Lambda \mathrm{J_s}) + (\mathrm{v_s}\cdot\nabla)\Lambda\mathrm{J_s} \quad \Leftarrow \text{Chain rule}$$

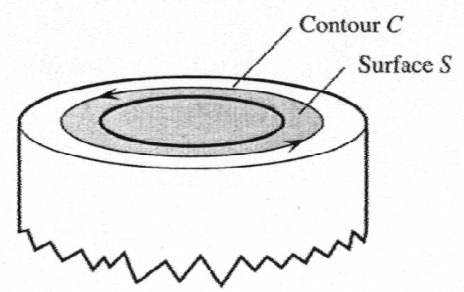
$$\frac{d}{dt}\left\{\Lambda n^* q^* \mathrm{v_s}\right\} = \mathrm{E} + (\mathrm{v_s}\times\mathrm{B}) \Leftarrow \Lambda = \frac{m^*}{n^*(q^*)^2}$$

$$\therefore \quad m^* \frac{d\mathrm{v_s}}{dt} = q^*\mathrm{E} + q^*(\mathrm{v_s}\times\mathrm{B})$$

5-5 Flux Quanta (Vortex)

✓ Take integration the supercurrent equation about closed contour *C*

$$\oint_C \left\{ \Lambda \mathbf{J}_S = -\left(\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right) \right\}$$



Applying the Stock's theorem we know

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{A})\cdot ds = \int_S \mathbf{B}\cdot ds$$

$$\oint_C \nabla\theta\cdot d\vec{l} = \lim_{\mathrm{r}_b\to\mathrm{r}_a}\left(\theta(\mathrm{r}_b)-\theta(\mathrm{r}_a)\right) = 2\pi n \Leftarrow \Psi(r) = \sqrt{\mathrm{n}^*}\, e^{i(\theta_p+2\pi n)}$$

where S is surface defined by the contour C and B is the magnetic flux density.

Therefore,

$$\boxed{\oint_C (\Lambda\mathbf{J}_S)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot ds = \frac{\hbar}{q^\star}2\pi \mathrm{n}}$$

n replace by -*n*

- The *flux quantum* is defined as

$$\boxed{\Phi_o \equiv \frac{2\pi\hbar}{|q^\star|} = \frac{h}{|q^\star|}}$$

positive quantity

- The *fluxoid quantization* condition

$$\underbrace{\oint_C (\Lambda\mathbf{J}_S)\cdot d\mathbf{l} + \int_S \mathbf{B}\cdot ds}_{fluxoid} = \mathrm{n}\Phi_o$$

from experiment one measured: $\Phi_o = \frac{h}{2e} = 2.07\times 10^{-15}\ \mathrm{T\text{-}m^2}$

SUMMARY

Extended concepts of superconductivity from Classical to Quantum theory

- The wave-particle duality of nature is explicit.
- The Schrodinger's equation $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi$ describes the dynamical revolution of the probability amplitude $\Psi(\mathrm{r}, t)$.
- The square of probability amplitude is *probability* indicate that the particle will be at specific place at time t.
- To describe the motion of this probability distribution we found the *probability current*.
- The Macroscopic wavefunction:

$$\boxed{\Psi(\mathbf{r},t) = \sqrt{n^\star}\, e^{i\theta(\mathbf{r},t)}}$$

- The macroscopic wavefunction obeys a Schrodingerlike equation:

$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \frac{1}{2m^\star}\left(\frac{\hbar}{i}\nabla - q^\star\mathbf{A}(\mathbf{r},t)\right)^2\Psi(\mathbf{r},t) + q^\star\phi(\mathbf{r},t)\,\Psi(\mathbf{r},t)}$$

- The MQM provides an extension of the concept of probability currents. Since describes may particles, the flow of probability for the entire ensemble is equivalent to the flow of the macroscopic supercurrent J_s as given below (in case of an isotropic supercurrent)

$$\boxed{\mathbf{J}_\mathrm{S} = q^\star n^\star(\mathbf{r},t)\left(\frac{\hbar}{m^\star}\nabla\theta(\mathbf{r},t) - \frac{q^\star}{m^\star}\mathbf{A}(\mathbf{r},t)\right)}$$

or

$$\boxed{\Lambda\mathbf{J}_\mathrm{S} = -\left(\mathbf{A}(\mathbf{r},t) - \frac{\hbar}{q^\star}\nabla\theta(\mathbf{r},t)\right)}$$

- Take time derivative and curl of the supercurrent will gives us the 1st and 2nd London eqn, respectively.

$$\boxed{\frac{\partial}{\partial t}(\Lambda\mathbf{J}_\mathrm{S}) = \mathbf{E} - \frac{1}{n^\star q^\star}\nabla\left(\frac{1}{2}\Lambda\mathbf{J}_\mathrm{S}{}^2\right)}$$

$$\boxed{\nabla\times(\Lambda\mathbf{J}_\mathrm{S}) = -\nabla\times\mathbf{A} = -\mathbf{B}}$$

- Finally, the fluxoid quantization can be obtained by integrating the supercurrent eqn around a path in a multiply connected region.

$$\boxed{\oint_C(\Lambda\mathbf{J}_\mathrm{S})\cdot d\mathbf{l} + \int_S\mathbf{B}\cdot d\mathbf{s} = \mathrm{n}\Phi_o}$$