Chap. 6 Type II Superconductivity

6.1 Introduction
6.2 The Vortex
6.3 The Modified Second London Equation
6.4 General Thermodynamic Concepts (skip)
6.5 Critical Fields
6.1 INTRODUCTION

In 1930: W.J. de Hass & J. Voogd – working with the superconducting alloy Pb–Bi.

The \( T_c \) (max) of this alloy (\( T_c = 8.8 \) K) does not differ much from its constituents.

(Lead \( T_c = 7.2 \) K, Bismuth \( T_c = 8.5 \) K)

But \( H_c \) (at 4.2K): \( H_c \) (Pb) = 0.055 T, \( H_c \) (Pb–Bi) = 1.7 T

Figure 6.1 The H–T phase space:
(a) type I and (b) type II superconductors.
★ This chapter develops the essential features of Abrikosov’s work through the phenomenology of type II superconductivity.

▶ In Section 6.2: We see that flux is observed to enter a type II superconductor in a discrete array of known as vortices.

Each vortex has a single flux quanta $\Phi_0$.

We will find the fields and currents associated with each vortex and show that the radius of the vortex (the coherence length $\xi$).

The two type superconductors can be distinguished by comparing $\lambda$ and $\xi$.

(type I superconductor – $\lambda \leq \xi$, type II superconductor – $\lambda \geq \xi$)

▶ In Section 6.3: We see that $B$ and $J$ from a vortex are more conveniently described by modifying the second London equation.
To fully understand vortex formation in type II materials

▶ In Section 6.4: We first review some basic concepts in equilibrium thermodynamics (skip).

▶ In Section 6.5: Section 6.4 concepts are then used to study the various critical magnetic fields.

★ We therefore see that this chapter focuses on understanding the vortices that give rise to the magnetic field phase boundaries, $H_{c1}(T)$ and $H_{c2}(T)$ for a type II superconductor.
6.2 THE VORTEX

Above the lower critical field, $H_{c1}(T)$, magnetic flux will enter a type II superconductor.

How we can detect the flux around a simple bar magnet

Magnetic field lines emanate primarily from the north pole of a magnet and curve around to the south pole.
Vortex and coherence length

Consider type II bulk superconductor ($a \gg \lambda$)

$$H_0 \rightarrow H_0 > H_{c1}$$

A vapor of tiny micron-sized nickel particles is now evaporated over the top surface of the superconductor, perpendicular to the applied field.

The particles will coalesce on the surface along lines of constant flux density just as in the case of the iron filings and thus the surface of the superconductor is “decorated”, showing how flux enters.
Figure 6.2 A type II superconductor that has been decorated with magnetic particle. The applied field is perpendicular to the page.

(a) A photograph revealing the triangular array of vortices formed when a superconducting sample, in this case $\text{YBa}_2\text{Cu}_3\text{O}_7$, is placed in a magnetic field. (The anisotropic sample is oriented so that the c axis is parallel to the applied field). The applied field strength is 40 Gauss and the distance 0.8$\mu$m.

(b) The triangular array of patterns; each patterns regions of constant flux density. $C_1$ is a contour along the perimeter of a pattern, $C_2$ is a contour around the center of one.
We see that the flux density is strongest at the center of each pattern. (the particles concentrate in regions of high-flux density)

The density of vortices $n_v$

$$n_v = \frac{1}{A} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)\alpha^2} = \frac{2}{\sqrt{3}\alpha^2} \quad (\alpha : \text{the distance between the centers of nearest patterns})$$

The average flux density in the slab

$$< B > = \Phi / A = n_v \Phi_0 \quad (\Phi_0 : \text{the flux in each vortex})$$

Since we can measure both $<B>$ and $\alpha$, the value of $\Phi_0$ is determined experimentally and found to always be single flux quantum ($h/2e$).
Let us find the consequences of this experimental fact on the flux and current densities in an isotropic type II SC.

From the fluxoid quantization condition (section 5.5)

\[
\Phi_{\text{tot}} = \int_C (\Lambda J_s) \cdot dl + \int_S B \cdot ds
\]  

— (6.3)

along the contour \(C_1\) (shown in Figure 6.2b)

\[
\Phi_0 = \int_{C_1} \mu_0 \lambda^2 J_s \cdot dl + \int_{S_1} B \cdot ds \quad (\lambda = \sqrt{\frac{\Lambda}{\mu_0}})
\]  

— (6.4)

The flux is greatest at center and so \(B_z(x,y)\) is at a minimum (constant) along the contour \(C_1\). Consequently,

\[
\frac{\partial}{\partial x} B_z(x_c, y_c) = \frac{\partial}{\partial y} B_z(x_c, y_c) = 0
\]  

— (6.5)

where \((x_c, y_c)\) are the loci of points that define the contour.
\[ \nabla \times \mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & B_z \end{vmatrix} = \left( \frac{\partial}{\partial y} B_z \right) \hat{i} - \left( \frac{\partial}{\partial x} B_z \right) \hat{k} \]

\( \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s \) and combining this relation with Eq. (6.5)

\[ \mathbf{J}_s = \frac{1}{\mu_0} \nabla \times \mathbf{B} = 0 \quad \text{along } C_1 \quad \text{— (6.6)} \]

Thus, the fluxoid quantization condition of Eq. (6.4) becomes

\[ \Phi_0 = \int_{S_1} \mathbf{B} \cdot d\mathbf{s} = BA = \left( \frac{\sqrt{3}}{2} \alpha^2 \right) B \quad \text{— (6.7)} \]

The surface \( S_1 \) is the area of one of the patterns so direct integration recovers the fact that

\[ <B> = \frac{N_v \Phi_0}{A} = n_v \Phi_0 \quad n_v : \text{vortex density} \quad \text{— (6.8)} \]
The flux quantization condition for the second contour $C_2$, which is a circle centered in one of the flux patterns

$$\Phi_0 = \oint_{C_2} \mu_0 \lambda^2 J_s \cdot dl + \int_{S_2} B \cdot ds \quad - (6.9)$$

In the limit that the radius $r$ of the circular contour $C_2$ approaches zero,

$$\Phi_0 = \lim_{r \to 0} \oint_{C_2} \mu_0 \lambda^2 J_s \cdot dl \quad \Leftarrow \lim_{r \to 0} \int_{S_2} B \cdot ds = \lim_{A \to 0} (BA) = 0 \quad - (6.10)$$

We expect $J_s$ to be constant along $C_2$ and azimuthally directed($\Phi$).

From (6.10)

$$\Phi_0 = \lim_{r \to 0} \mu_0 \lambda^2 J_s \cdot \oint_{C_2} dl = \lim_{r \to 0} \mu_0 \lambda^2 J_s (2\pi r)$$

$$\lim_{r \to 0} J_s = \frac{\Phi_0}{2\pi \mu_0 \lambda^2} \frac{1}{r} \sin \phi \quad - (6.11)$$

In Eq. (6.11)

$$J_s \propto \frac{1}{r} \quad \Rightarrow \quad r \to 0 \Rightarrow \quad J_s \to \infty \quad \text{Not physical !!!}$$
How we can overcome this problem??

We can overcome this problem by choosing to center of the vortex as having a normal core region.

If core has a radius $\xi$, then the maximum supercurrent density is

$$J_{s}^{\text{max}} = \frac{\Phi_0}{2\pi \mu_0 \lambda^2} \frac{1}{\xi} i_{\phi}$$

--- (6.12)

a bounded expression.

$\xi$ is the radius of the normal region, we find the increase in the energy $\delta \varepsilon$ of each of the Cooper pairs (the gap energy $2\Delta$) as the core is approached.
Velocity of the Cooper pairs

The increase in energy comes from the increase in the kinetic energy of the Cooper pairs. To see this, recall

\[ J_s = n^* q^* v_s \quad \text{— (6.13)} \]

\( v_s \) : the velocity of the Cooper pairs (superelectrons)

The maximum velocity (from Eq. (6.12))

\[ J_s^{\text{max}} = \frac{\Phi_0}{2\pi\mu_0\lambda^2} \frac{1}{\xi} i_\phi \quad \text{— (6.12)} \]

\[ v_{s}^{\text{max}} = \frac{\hbar}{m^*} \frac{1}{\xi} i_\phi \quad \text{See next page} \quad \text{— (6.14)} \]

The increase in the energy of a Cooper pair is created by the increase in its velocity.

If there are no currents flowing, the electrons are all moving with some background velocity known as the Fermi velocity \( v_F \) (in Section 5.4).
<Calculation of supercurrent velocity>

\[ J_s = \hbar q^* v_s = \frac{\Phi_0}{2\pi\mu_0\lambda^2} \frac{1}{\xi} i^\phi = \frac{\Phi_0}{2\pi \Lambda \xi} i^\phi \ (\leftarrow \lambda = \sqrt{\frac{\Lambda}{\mu_0}}) \]

\[ = \frac{\Phi_0}{2\pi \frac{m^*}{\xi}} \frac{1}{\xi} i^\phi = \frac{\Phi_0 \hbar (q^*)^2}{2\pi m^* \frac{\xi}{\xi}} i^\phi \]

\[ v_s = \frac{\Phi_0 \frac{q^*}{2e}}{2\pi m^* \frac{\xi}{\xi}} i^\phi = \frac{\hbar}{2e} \frac{q^*}{2\pi m^* \frac{\xi}{\xi}} i^\phi = \frac{\hbar q^*}{2\pi m^* \frac{\xi}{\xi}} i^\phi \]

\[ = \frac{\hbar}{2\pi m^* \frac{\xi}{\xi}} i^\phi = \frac{2\pi}{\hbar} \frac{1}{2\pi m^* \frac{\xi}{\xi}} i^\phi = \frac{\hbar}{m^* \xi} i^\phi \]

\[ \therefore v_s = \frac{\hbar}{m^* \xi} i^\phi \]

\[ v_s = \frac{10^{-34} J}{1.82 \times 10^{-30} Kg \ 3 \times 10^{-9} m} \sim 10^4 m/s \] for YBCO.
Superconducting Energy Gap (2Δ) and Coherence length

The average kinetic energy of a Cooper pair when no current is flowing:

$$\varepsilon_{\text{kin}}^0 = \frac{1}{2} m^* v_F^2 = \frac{1}{2} m^* (v_{F,x}^2 + v_{F,y}^2 + v_{F,z}^2)$$  

— (6.15)

=> Now consider the increased velocity due to the flow of current (J_y).

$$\varepsilon_{\text{kin}}^1 = \frac{1}{2} m^* \left[ v_{F,x}^2 + \left( v_{F,y} + v_{s,\phi}^{\text{max}} \right)^2 + v_{F,z}^2 \right]$$  

— (6.16)

The difference in energy at the core

$$\delta \varepsilon = \varepsilon_{\text{kin}}^1 - \varepsilon_{\text{kin}}^0$$

— (6.17)

The additional velocity created by the current is much less than the Fermi velocity so the linearizing we find ($v_{F,y} \gg v_{s,\phi}^{\text{max}}$)

$$\Rightarrow \delta \varepsilon \approx m^* v_{F,y} v_{s,\phi}^{\text{max}}$$  

— (6.18)
\[ \delta \varepsilon \approx m^* v_{F,y} v_{s,\phi} \] ----- (6-18)

We expect that by averaging over all the electrons, we will find
\[ v_F^2 =< v_{F,x}^2 > + < v_{F,y}^2 > + < v_{F,z}^2 > \]
(\text{where the bracket denotes the average of the component of the velocity.})

\[ \text{The averaged velocity squared to be the same for each component} \]

\[ < v_{F,y}^2 > = \frac{1}{3} v_F^2 \]

--- (6.20)

Substituting this averaged expression for \( v_{F,y} \) and our relation for \( v_{s,\phi} \max \) (Eq. 6.14) into Eq. 6.18 gives
\[ \delta \varepsilon \approx m^* v_{F,y} v_{s,\phi} \max = m^* \frac{1}{\sqrt{3}} v_F \frac{\hbar}{m^*} \frac{1}{\xi} \Rightarrow \delta \varepsilon \approx \frac{\hbar v_F}{\sqrt{3} \xi} \]

--- (6.21)

Since \( \delta \varepsilon \approx 2\Delta \) at the radius of the core \( \xi \), we find
\[ \xi \approx \frac{\hbar v_F}{2\sqrt{3} \Delta} \]

--- (6.22)
At zero temperature the energy gap is $2\Delta_0$ and so the corresponding core radius, denoted $\xi_0$, is given by

$$
\xi_0 \approx \frac{\hbar v_F}{2\sqrt{3}\Delta_0} = \frac{1}{3.46} \frac{\hbar v_F}{\Delta_0}
$$

— (6.23)

From the microscopic BCS theory

$$
\xi_0 = \frac{\hbar v_F}{\pi \Delta_0}
$$

— (6.24)

(where $\xi_0$ is known as the BCS coherence length)

Near the $T_c$, the BCS theory shows that the energy gap goes to zero as

$$
\Delta \propto \sqrt{1 - \left(\frac{T}{T_c}\right)}
$$

Therefore,

$$
\lim_{T \to T_c} \xi(T) = \frac{\xi(0)}{\sqrt{1 - \left(\frac{T}{T_c}\right)}}
$$

— (6.25)

(where $\xi(0)$ is the temperature independent coefficient of $\xi$ near $T_c$)
Temperature dependence of coherence length

\[
\lim_{T \to T_c} \frac{\xi(T)}{\xi(0)} = \frac{\xi(0)}{\sqrt{1 - (T / T_c)}}
\]
We have found the coherence length by arguing that the Cooper pairs split apart of depair when their increase in energy exceeds the gap energy. Thus the maximum current density for $J_{s,\Phi}$ (Eq. 6.12) is referred to as the depairing critical current density $J_{\text{pair}}$. Therefore,

$$ J_{\text{pair}} \approx \frac{\Phi_0}{2\pi\mu_0 \lambda^2 \xi} \quad \text{from} \quad J_s = \frac{\Phi_0}{2\pi\mu_0 \lambda^2} \frac{1}{r} $$

This is the maximum current density that can be sent through the superconductor. This expression is only approximate because it depends on our model of the vortex core.

by the Ginzburg–Landau theory,

$$ J_{\text{pair}} = \frac{\Phi_0}{3\sqrt{3}\pi\mu_0 \lambda^2 \xi} = \frac{1}{5.2} \frac{\Phi_0}{\pi\mu_0 \lambda^2 \xi} \quad \text{— (6.27)} $$

If we apply a current density, $J > J_{\text{pair}}$, the material becomes normal.
※ Summary

- The magnetic flux enters a type II superconductor for fields above $H_{c1}$ in a regular triangular array of vortices. (experiment)
- Each vortex has a single flux quantum.
- The current density of the vortex increases as the center of the vortex is approached until the increased kinetic energy causes the Cooper pairs to unbind.
- The vortex is thus modeled as having a normal core of radius $\xi$, the coherence length.
Figure 6.3 shows our model of the density of superelectrons (Cooper pairs) for a single vortex. Inside the model core the conduction electrons are in the normal state.
What makes our approach reasonable is the fact that most practical type II superconductors have $\xi \ll \lambda$. Thus, the core typically occupies a very small fraction of the total volume of the vortex. In the Ginzburg–Landau theory, the ratio of the two lengths is defined as

$$\kappa \equiv \frac{\lambda}{\xi}$$

(\(\kappa\): Ginzburg–Landau kappa)

- $\xi \geq \lambda$: Type-I Superconductor
- $\xi \leq \lambda$: Type-II Superconductor
- $\xi \ll \lambda$: High-$\kappa$ Materials
6.3 THE MODIFIED SECOND LONDON EQUATION

* Purpose: With vortex–core model for how flux distributes itself in a type II superconductor, we will find the full spatial dependence of the flux density and currents associated with the vortex.

In Figure 6.3, has a normal cylindrical core with a radius $\xi$. In the superconducting region outside the core, the local currents and fields are governed by the 2nd London equation.

$$\nabla \times \left( \Lambda J_s \right) + \mathbf{B} = 0 \quad \text{— (6.29)}$$

Outside the normal core the magnetic flux density satisfies the vector Helmholtz equation

$$\nabla^2 \mathbf{B}(r) - \frac{1}{\lambda^2} \mathbf{B}(r) = 0 \quad (\text{for } r \geq \xi) \quad \text{— (6.30)}$$

Because of the cylindrical symmetry of the vortex structure with its core centered along the z-axis, $\mathbf{B} = B_z(\mathbf{r}) \mathbf{i}_z$, where $\mathbf{r}$ is the radius vector in cylindrical coordinates.

$$\nabla^2 B_z - \frac{1}{\lambda^2} B_z = 0 \quad (\text{for } r \geq \xi) \quad \text{— (6.31)}$$
The solution to Eq. (6.31)

\[ B_z (r, \phi) = \sum_{m=0}^{\infty} K_m \left( \frac{r}{\lambda} \right) \left( C_m \cos m \phi + C'_m \sin m \phi \right) \]

\[ + \sum_{m=0}^{\infty} I_m \left( \frac{r}{\lambda} \right) \left( D_m \cos m \phi + D'_m \sin m \phi \right) \]

\[-\cdots-(6-32)\]

* C_m, C'_m, D_m, D'_m: constant
* I_m: modified Bessel functions of the first order with m
* K_m: modified Bessel functions of the second order with m

Modified Bessel functions

**For m = 0**

\[ \lim_{x \to 0} I_0(x) \to 1, \quad \lim_{x \to 0} K_0(x) \to -\ln x \]

**For m > 0**

\[ \lim_{x \to 0} I_m(x) \to \frac{1}{m!} \left(\frac{x}{2}\right)^m \]

\[ \lim_{x \to 0} K_m(x) \to \frac{1}{2m!} \left(\frac{2}{x}\right)^m \]
**Bz** is independent of the angle $\Phi$ for the flux of the single vortex. Therefore, the solution is obtained with $m = 0$

$$B_z(r, \phi) = \sum_{m=0}^{\infty} K_m \left( \frac{r}{\lambda} \right) (C_m \cos m\phi + C_m' \sin m\phi) + \sum_{m=0}^{\infty} I_m \left( \frac{r}{\lambda} \right) (D_m \cos m\phi + D_m' \sin m\phi)$$

(6.33)

(1) We find the values of $C_0$ & $D_0$ in BC.

* $r \to \infty \implies B_z \to 0$

* $D_0 = 0$ because $I_0 \to \infty$ for $r \to \infty$

(2) The other BC at the radius of the core:

The core is normal and $B_z$ is constant for $r < \xi$

$$B_z(r) = \begin{cases} C_0K_0 \left( \frac{\xi}{\lambda} \right) & \text{for } r < \xi \\ C_0K_0 \left( \frac{r}{\lambda} \right) & \text{for } r \geq \xi \end{cases}$$

----- (6.34)
The coefficient $C_0$ is determined by applying the fluxoid quantization condition. In section 6.2

$$\Phi_{\text{tot}} = \oint_C (\Lambda J_s) \cdot dl + \int_S \mathbf{B} \cdot d\mathbf{s} = \Phi_0 \quad - (6.35)$$

Let the contour of integration be a circle of radius $R_c$ in the $x$-y plane.

$B_z$ decays exponentially to zero for $r \gg \lambda$

→ the associated current density will also decay in the same manner.

Consequently,

As $R_c \to \infty$, $J_s \to 0$, the first term in Eq. (6.35) → 0

The only contribution to the fluxoid quantization condition comes from the surface integral term where the chosen contour is the entire $x$-y plane.
$C_0$ can be determined by the Flux Quantization condition.

$$B_z(r) = \begin{cases} 
  C_0 K_0(\frac{r}{\lambda}) & \text{for } r \geq \xi \\
  C_0 K_0(\frac{\xi}{\lambda}) & \text{for } r < \xi 
\end{cases}$$

$$\Phi_0 = \int_s B \cdot ds = \int_s B_z \cdot ds = \int_0^\xi C_0 K_0(\frac{\xi}{\lambda}) 2\pi r \, dr + \int_\xi^\infty C_0 K_0(\frac{r}{\lambda}) 2\pi r \, dr$$

$$= \int_0^\xi C_0 K_0(\frac{\xi}{\lambda}) 2\pi r \, dr + \int_\xi^\infty C_0 2\pi \lambda \frac{r}{\lambda} K_0(\frac{r}{\lambda}) d\left(\frac{r}{\lambda}\right) \lambda \leftrightarrow dr \leftrightarrow dx$$

$$= C_0 K_0(\frac{\xi}{\lambda}) 2\pi \left[ \frac{1}{2} r^2 \right]_0^\xi + C_0 2\pi \lambda^2 (-\frac{r}{\lambda}) K_1(\frac{r}{\lambda}) \bigg|_\xi^\infty$$

$$= C_0 K_0(\frac{\xi}{\lambda}) \pi \xi^2 + C_0 2\pi \lambda^2 \frac{\xi}{\lambda} K_1(\frac{\xi}{\lambda})$$

$$= C_0 2\pi \lambda^2 \left[ \frac{1}{2}\frac{\xi^2}{\lambda^2} K_0(\frac{\xi}{\lambda}) + \frac{\xi}{\lambda} K_1(\frac{\xi}{\lambda}) \right]$$

$$\therefore C_0 = \frac{\Phi_0}{2\pi \lambda^2} \left[ \frac{1}{2}\frac{\xi^2}{\lambda^2} K_0(\frac{\xi}{\lambda}) + \frac{\xi}{\lambda} K_1(\frac{\xi}{\lambda}) \right]^{-1} \quad (6.37)$$
A considerable simplification is possible of most practical type II materials [High-κ materials (κ = λ/ξ ⩾ 1)]

Ex) Nb₃Sn κ ≈ 25 & HTS typically have κ > 50

Consider how C₀ simplifies for κ ≫ 1.

\[
C₀ = \frac{\Phi₀}{2\pi\lambda^2} \left[ \frac{1}{2} \frac{\xi^2}{\lambda^2} K₀ \left( \frac{\xi}{\lambda} \right) + \frac{\xi}{\lambda} K₁ \left( \frac{\xi}{\lambda} \right) \right] \approx \frac{\Phi₀}{2\pi\lambda^2} \quad \text{(for κ≫1)}
\]

The first term

\[
\lim_{x \to 0} x^2 K₀(x) = \lim_{x \to 0} x^2 \ln x = 0 \quad \text{(for κ≫1)}
\]

The second term

\[
\lim_{x \to 0} xK₁(x) = 1
\]

Therefore, for κ ≫ 1, (from Eq.(6.34))

\[
B_z(r) = \begin{cases} 
C₀K₀ \left( \frac{r}{\lambda} \right) & \text{for } r \geq \xi \\
C₀K₀ \left( \frac{\xi}{\lambda} \right) & \text{for } r < \xi 
\end{cases}
\]

\[
B(r) = \begin{cases} 
\frac{\Phi₀}{2\pi\lambda^2} K₀ \left( \frac{r}{\lambda} \right) i_z & \text{for } r \geq \xi \\
\frac{\Phi₀}{2\pi\lambda^2} K₀ \left( \frac{\xi}{\lambda} \right) i_z & \text{for } r < \xi
\end{cases}
\]
The \( J_s(r) \) associated with \( B(r) \), can be found from Ampère’s law (\( \nabla \times B = \mu_0 J_s \))

\[
\nabla = i_r \frac{\partial}{\partial r} + i_\phi \frac{1}{r} \frac{\partial}{\partial \phi} + i_z \frac{\partial}{\partial z}
\]

for cylindrical coordinates

\[
\frac{r}{\lambda} = x, \quad \frac{dr}{\lambda} = dx, \quad \frac{dK_0(x)}{dr} = \frac{1}{\lambda} K_0'(x), \quad K_0'(x) = -K_1(x), \quad i_r \times i_z = -i_\phi
\]

\[
B(r) = \begin{cases} 
\frac{\Phi_0}{2\pi\lambda^2} K_0 \left( \frac{r}{\lambda} \right) i_z & \text{for } r \geq \xi \\
\frac{\Phi_0}{2\pi\lambda^2} K_0 \left( \frac{\xi}{\lambda} \right) i_z & \text{for } r < \xi
\end{cases}
\]

\[
J_S(r) = \begin{cases} 
\frac{\Phi_0}{2\pi\mu_0\lambda^3} K_1 \left( \frac{r}{\lambda} \right) i_\phi & \text{for } r \geq \xi \\
0 & \text{for } r < \xi
\end{cases}
\]  

(6.41)

Figure 6.5 shows how both the flux and current density decay exponentially far from the center of the vortex.
Near the core ($\xi \leq r \ll \lambda$), the magnetic flux density ($B$) in the superconductor increases as:

$$B = \frac{\Phi_0}{2\pi\lambda^2} K_0 \left(\frac{r}{\lambda}\right) i_z \quad (\text{for } r \geq \xi) \leftarrow \lim_{x \to 0} K_0 \left(\frac{r}{\lambda}\right) \rightarrow -\ln\left(\frac{r}{\lambda}\right)$$

and current density ($J_s$) is given by:

$$\lim_{\xi \leq r \ll \lambda} B = \frac{\Phi_0}{2\pi\lambda^2} \ln \frac{\lambda}{r} i_z \quad \text{—— (6.42)}$$

$$\lim_{\xi \leq r \ll \lambda} J_s = \lim_{\xi \leq r \ll \lambda} \left(\nabla \times B / \mu_0\right) = \frac{\Phi_0}{2\pi\mu_0 \lambda^2} \frac{1}{r} i_\phi \quad \text{—— (6.43)}$$

The expression for the current density near the core is identical to Eq. (6.11), which was obtained by considering the fluxoid quantization.

$$\lim_{r \to 0} J_s = \frac{\Phi_0}{2\pi\mu_0 \lambda^2} \frac{1}{r} i_\phi \quad \text{—— (6.11)}$$
We seek to modify the second London equation by adding a function $V(r)$ as a source to insure fluxoid conservation.

\[
\nabla \times (\Lambda J_s) + B = V(r) \quad \text{Modified second London equation} \quad \text{-- (6.44)}
\]

which is valid for all space assuming $V(r)$ vanishes except along $i_z$.

$V(r)$ is in the same direction as the flux density

\[
V(r) = V(r) i_z \quad \text{-- (6.45)}
\]

Eq (6.44) is converted into the integral expression

\[
\int_S \nabla \times (\Lambda J_s) \cdot ds + \int_S B \cdot ds = \int_S V(r) \cdot ds
\]

(by stoke’s theorem)

\[
\Rightarrow \oint_C (\Lambda J_s) \cdot dl + \int_S B \cdot ds = \int_S V(r) \cdot ds \quad \text{-- (6.46)}
\]
\[
\oint_C (\Lambda J_s) \cdot dl + \oint_S B \cdot ds = \Phi_0
\] — (6.35)

\[
\oint_C (\Lambda J_s) \cdot dl + \oint_S B \cdot ds = \int_S V(r) \cdot ds
\] — (6.46)

\( R_c \to \infty, J_s \to 0: \) the first terms in Eq(6.35) & (6.46) \( \to 0 \)

We see that although \( V(r) \) is zero everywhere except at \( r = 0 \), its integral over that point must be the constant \( \Phi_0 \).

The only function that has this property is the two-dimensional delta function, \( \delta_2(r) \).

\[
V(r) = \Phi_0 \delta_2(r) i_z \quad \leftarrow \text{Vorticity} \] — (6.47)

\[
\nabla \times (\Lambda J_s) + B = \Phi_0 \delta_2(r) i_z
\] — (6.48)

(for a vortex along the \( z \)-axis in an isotropic, high-\( \kappa \) superconductor)

More generally, for \( N \) vortices that located at the two-dimensional positions \( r_i \), the vorticity is readily generalized to

\[
V(r) = \sum_{i=1}^{N} \Phi_0 \delta_2(r - r_i) i_z
\] — (6.49)
Example 6.3.1

As an example of how to use the modified second London equation, we complete the solution of our single vortex problem. For an isotropic superconductor, from Eq. (6.48)

$$\nabla^2 B_z - \frac{1}{\lambda^2} B_z = -\frac{\Phi_0}{\lambda^2} \delta_2(r)$$  \hspace{1cm} (6.50)

Particular solution, $B_z^p$

$$B_z^p = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right)$$  \hspace{1cm} (6.51)

This solution not only satisfies the original differential equation but also the boundary condition that the flux density vanishes far from the core. Moreover, this solution also satisfies the fluxiod quantization condition for a high-$\kappa$ superconductor by construction.

Therefore, $B_z^p$ is the desired flux density distribution for the single vortex and the associated current density is

$$J_{s,\phi} = \frac{1}{\mu_0} (\nabla \times B_z^p) = \frac{\Phi_0}{2\pi\mu_0\lambda^2} K_1\left(\frac{r}{\lambda}\right)$$  \hspace{1cm} (6.52)
Derivation of Equ. (6.50)

\[ \nabla \times (\Lambda J_s) + B = \Phi_0 \delta_2(r) \mathbf{i}_z \quad \text{---- (6-48)} \]

\[ \nabla \times H = J \]

Take a curl (\(\nabla \times\)) for both side;

\[ \nabla \times J = \nabla \times \nabla \times H = \nabla(\nabla \circ H) - \nabla^2 H = -\nabla^2 H \]

\[ \nabla \times (\Lambda J_s) = -\Lambda \nabla^2 H = -\mu_0 \lambda^2 \nabla^2 \frac{1}{\mu_0} B = -\lambda^2 \nabla^2 B \]

\[ (\because \lambda = \sqrt{\Lambda / \mu_0} \Rightarrow \Lambda = \mu_0 \lambda^2) \]

\[ -\lambda^2 \nabla^2 B + B = \Phi_0 \delta_2(r) \mathbf{i}_z \]

\[ \therefore \nabla^2 B_z - \frac{1}{\lambda^2} B_z = -\frac{\Phi_0}{\lambda^2} \delta_2(r) \]
\( (6.40) \) \quad B(r) = \begin{cases} \frac{\Phi_0}{2\pi\lambda^2} K_0 \left( \frac{r}{\lambda} \right) i_z & \text{for } r \geq \xi \\ \frac{\Phi_0}{2\pi\lambda^2} K_0 \left( \frac{\xi}{\lambda} \right) i_z & \text{for } r < \xi \end{cases} \quad - (6.41)

\( (6.51) \quad B^p_z = \frac{\Phi_0}{2\pi\lambda^2} K_0 \left( \frac{r}{\lambda} \right) \quad J_{s,\phi} = \frac{\Phi_0}{2\pi\mu_0\lambda^3} K_1 \left( \frac{r}{\lambda} \right) \quad - (6.52) \)

These solutions match those of the finite-sized vortex core in the region \( r \geq \xi \).

However, for \( r < \xi \), the solutions differ.

\[ \lim_{{x \to 0}} K_0(x) \to -\ln x \quad \lim_{{x \to 0}} K_1(x) \to \frac{1}{x} \]

Eq(6.51) & Eq(6.52) diverge as \( r \to 0 \). \quad \leftarrow \text{unphysical}!!!

Consequently, to ensure that the solutions of the modified second London equation match the physically correct ones, we restrict the validity of the solutions to the region \( r \geq \xi \). Within the region \( r \leq \xi \), we will interpret the flux density to be a constant defined by its value at \( r = \xi \).
Example 6.3.2

Consider now the solution to the modified second London equation for an array of vortices defined by Eq(6.49). For an isotropic superconductor that fills all of space, Eq(6.44) leads to

\[ \nabla \times (\Lambda J_s) + B = V(r) \quad - (6.44) \]
\[ V(r) = \sum_i \Phi_0 \delta_2 (r - r_i) \hat{i}_z \quad - (6.49) \]

\[ \nabla^2 B_z - \frac{1}{\lambda^2} B_z = -\sum_i \frac{\Phi_0}{2} \delta_2 (r - r_i) \quad - (6.53) \]

Since Eq(6.53) is linear, the solution is just the superposition of the particular solution for a single vortex \( B_z^i \) centered at each array site; namely,

\[ B_z (r) = \sum_p \frac{\Phi_0}{2 \pi \lambda^2} K_0 \left( \frac{|r - r_p|}{\lambda} \right) \quad - (6.54) \]
Example 6.3.4

We now examine the case of two vortices that individually generate a magnetic flux density in opposite directions.

\[ V(r) = \Phi_0 \delta(x - a)\delta(y) i_z - \Phi_0 \delta(x + a)\delta(y) i_z \quad (6.59) \]

\[ B_z'(\frac{r}{a}) \Rightarrow \quad B_z(x, y) = \frac{\Phi_0}{2\pi\lambda^2} \left[ K_0 \left( \frac{\sqrt{(x-a)^2+y^2}}{\lambda} \right) - K_0 \left( \frac{\sqrt{(x+a)^2+y^2}}{\lambda} \right) \right] \quad (6.60) \]

The flux density vanishes along the y–z plane defined by x=0 (antisymmetry). However, the current density does not vanish in this plane. Instead, it is purely y-directed.

Figure 6.9 Two vortices parallel to the z-axis and located along the x-axis at ±a in an infinite superconductor. The vortex at x=a has a positive vorticity and the other at x=-a has a negative one. The contours of constant \( B_z(x,y) \) are shown.
Vortex Energy ($\varepsilon_v$)

We end this section by showing how it can also simplify finding the electromagnetic energy associated with vortices.

If the vortices are modeled as having a normal state core, the total electromagnetic energy $W$ of the system will come from three places:

* $W_s$: the energy of superconducting region.
* $W_c$: the energy of the normal cores in the superconductor.
* $W_n$: the energy of the normal regions (no superconductor).

Integrating over the volume where there is no superconductor, $V_n$, we find

\[
W_n = \frac{1}{2\mu_0} \int_{V_n} B^2 \, dv
\]

\[\text{Magnetic energy density} \quad E_m = \frac{1}{2} H \cdot B \]

\[
W_c = \frac{1}{2\mu_0} \int_{V_{\text{core}}} B^2 \, dv
\]

where the integration is over the cores of the vortices & high-$\kappa$ superconductors ($\lambda \gg \xi$), the contribution to the energy from the normal core is negligible ($W_c \approx 0$).
From 4–130, \[
W_s = \frac{1}{2\mu_0} \int_{V_s} [B^2 + (\mu_0 J_s) \cdot (\Lambda J_s)] \, dv
\] — (6.66)

Using the Ampère’s law \((\nabla \times H = J_s)\), Eq(6.66) becomes

\[
W_s = \frac{1}{2\mu_0} \int_{V_s} [B^2 + (\nabla \times B) \cdot (\Lambda J_s)] \, dv
\] — (6.67)

The vector identity

\[
D \cdot (\nabla \times C) = C \cdot (\nabla \times D) + \nabla \cdot (C \times D)
\] — (6.68)

\[
W_s = \frac{1}{2\mu_0} \int_{V_s} [B^2 + B \cdot (\nabla \times (\Lambda J_s))] \, dv + \frac{1}{2\mu_0} \int_{V_s} \nabla \cdot [B \times (\Lambda J_s)] \, dv
\]

\[
= \frac{1}{2\mu_0} \int_{V_s} B \cdot [B + \nabla \times (\Lambda J_s)] \, dv + \frac{1}{2\mu_0} \int_{\Sigma_s} [B \times (\Lambda J_s)] \cdot ds
\] — (6.69)

where the integral over the surface \(\Sigma_s\) that encloses \(V_s\) is obtained by applying Gauss’ theorem.

by the modified second London equation \(\nabla \times (\Lambda J_s) + B = V(r)\)

\[
W_s = \frac{1}{2\mu_0} \int_{V_s} B \cdot V \, dv + \frac{1}{2\mu_0} \int_{\Sigma_s} [B \times (\Lambda J_s)] \cdot ds
\] — (6.70)
The energy per unit length along the vortex direction will be denoted as and similarly for each contribution $W_n'$ and $W_s'$.

$$W' = \frac{W}{L_z}$$  \hspace{1cm} (6.71)

(where $L_z$ is the length of the vortex in the z-direction.)

Therefore, after the integrating Eq(6.70) over $z$

$$W_s' = \frac{1}{2\mu_0} \int_{S_s} B \cdot V \, da + \frac{1}{2\mu_0} \oint_{C_s} [B \times (\Lambda J_s)] \cdot dh$$  \hspace{1cm} (6.72)

$$\Rightarrow da = dx dy$$

\(\Rightarrow\) Figure 6.11 A vortex along the z–direction in a superconducting volume $V_s$. The volume is enclosed by the surface $\Sigma_s$. The cross-sectional area $S_s$ is parallel to planes of constant z and is encircled by the contour $C_s$. Notice that $S_s$ is not a function of z.
Let us use Eq. (6.72) to find $\varepsilon_V$ (the electromagnetic energy per unit length for a single vortex in an infinite isotropic superconductor).

In Example 6.3.1, particular solution (the flux density) given as

$$B_z = B_z^p = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right)$$ — (6.73)

Both the flux density and the current decay exponentially from the vortex core. Thus the second term in Eq (6.72) can be made vanishingly small if the contour $C_s$ is taken far from the core.

$$W'_s = \frac{1}{2\mu_0} \int_{S_s} B \cdot V \, da + \frac{1}{2\mu_0} \oint_{C_s} [B \times (\nabla J_s)] \cdot dn$$ — (6.72)

Vortex energy per unit length with $V(r) \equiv \Phi_0 \delta_2(r) \mathbf{i}_z = \Phi_0 \mathbf{i}_z \quad \mathbf{a} \mathbf{r} = \xi$

$$\varepsilon_V = \frac{1}{2\mu_0} \int_{S_s} \left( \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right) \mathbf{i}_z \cdot \Phi_0 \delta_2(r) \mathbf{i}_z \right) \, da = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} K_0\left(\frac{\xi}{\lambda}\right)$$ — (6.74)
For the high $k$ materials:

\[
\lim_{x \to 0} K_0(x) \to -\ln x \\
\lim_{\lambda \gg \xi} \varepsilon_V = \frac{\Phi_0^2}{4\pi \mu_0 \lambda^2} \ln\left(\frac{\lambda}{\xi}\right)
\] — (6.75)
6.5 Critical Fields (H_{c1} and H_{c2})

※ Purpose: we calculate the upper [H_{c2}(T)] and lower [H_{c1}(T)] critical fields for a type II superconductor where we will need to combine the thermodynamics of a superconductor developed in section 6.4.

We restrict ourselves to examining the properties of an isotropic superconducting slab of thickness 2a where a \gg \lambda.

Gibbs free energy in a magnetic field is given by:

\[ G(\vec{H}, T) = -TS(\vec{H}, T) + U(\vec{H}, T) - V \vec{H} \cdot \vec{B} \]  \quad (6.121)

where U is internal energy and V is volume.

Without applied field (H = 0), the free energy is given by:

\[ G(0, T) = -TS(0, T) + U(0, T) \]  \quad (6.122)
The increase in the internal energy ($U$) due to the presence of the field is the quantity $W$ that was found in Section 6.3:

$$U(\vec{H}, T) = U(0, T) + W$$  — (6.123)

Combining Eq. (6.121) through (6.123)

$$G(0, T) = -TS(0, T) + U(0, T) \Rightarrow U(0, T) = G(0, T) + TS(0, T) \text{ from (6.121)}$$

$$\therefore \ U(\vec{H}, T) = G(0, T) + TS(0, T) + W \text{ from (6.123)}$$

(6.121) $\Rightarrow$ \[
G(\vec{H}, T) = -TS(\vec{H}, T) + G(0, T) + TS(0, T) + W - V \vec{H} \cdot \vec{B} \\
= G(0, T) + W - V \vec{H} \cdot \vec{B} - TS(\vec{H}, T) + TS(0, T) \\
= G(0, T) + W - V \vec{H} \cdot \vec{B} - T[S(\vec{H}, T) - S(0, T)]
\]  — (6.124)

The last term in this expression is just the heat exchange in the sample if the field is generated $\Delta Q = T\Delta S$.

For the second-order phase transition that no heat is generated whenever we consider the phase boundaries like $H_{c1}$ and $H_{c2}$ for a type II superconductor.

$$\Delta Q = T\Delta S(\vec{H}, T) = 0$$
We can therefore rewrite Eq(6.124) as

\[ G(\mathbf{H}, T) = G(0, T) + W - V \mathbf{H} \cdot \mathbf{B} \quad - (6.126) \]

The free energy in terms of the flux density B:

\[ G(\mathbf{H}, T) = G(0, T) + W - \mathbf{H} \cdot \int_V \mathbf{B} \, dv \quad - (6.128) \]

With \( W \) from Eq. (6.66: \( W_s = \frac{1}{2\mu_0} \int_V [B^2 + \mu_0 J_s \cdot (\Lambda J_s)] \, dv \)) and the superconducting current density given as \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s \) the Gibbs free energy for a superconductor is

\[
G_s(\mathbf{H}, T) = G_s(0, T) + \frac{1}{2\mu_0} \int_{V_s} \left[ \mathbf{B}^2 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \cdot (\Lambda \nabla \times \mathbf{B}) \right] \, dv
\]

\[ - \mathbf{H} \cdot \int_{V_s} \mathbf{B} \, dv \quad - (6.129) \]
Likewise, for the normal material volume \((J_s = 0)\), \(V_n\),

\[
G_n(\vec{H}, T) = G_n(0, T) + \frac{1}{2\mu_0} \int_{V_n} B^2 \, dv - \vec{H} \cdot \int_{V_n} \vec{B} \, dv
\]

\[\text{--- (6.130)}\]

The lower critical field is found by considering the difference in the Gibbs free energy \(G(\vec{H}, T)\) of the superconductor when the vortex is absent and when it is present.

Let us consider type II superconducting slab, shown in Figure 6.14.

When there are no vortices \(B \approx 0\).

\[
G^0_s(\vec{H}, T) = G^0_s(0, T)
\]

\[\text{--- (6.131)}\]

[Figure 6.14 Field distribution in a type II superconductor \((a \gg \lambda)\).]
When one vortex is in the center of the slab the corresponding Gibbs free energy \( G_s^1(\vec{H}, T) \) is

\[
G_s^1(\vec{H}, T) = G_s^0(0, T) + W_s - \vec{H} \cdot \int_{V_s} B \, dv
\]  

— (6.132)

In Section 6.3 the electromagnetic energy of a single vortex of length \( L_z \) was given by

\[
W_s = \varepsilon_V L_z
\]  

— (6.134)

\( \varepsilon_V \) : the energy per unit length of the vortex

The thermodynamic field \( \vec{H} \) for the slab is the applied field and so

\[
\vec{H} = H_{\text{app}}
\]  

— (6.135)
For a single vortex, we find that

\[ G_1^s(\overrightarrow{H}, T) = G_0^s(0, T) + W_s - \overrightarrow{H} \cdot \int_{V_s} B dv \quad \text{--- (6.132)} \]

Putting these expressions into Eq. (6-132) together yields the Gibbs free energy of the slab containing a single vortex

\[ G_1^s(\overrightarrow{H}, T) = G_0^s(0, T) + \varepsilon_v L_z - \overrightarrow{H} \Phi_0 L_z \quad \text{--- (6.137)} \]

The difference in Gibbs free energies between having and not having a single vortex present follows from Eq(6.137) and (6.131) :

\[ G_1^s(\overrightarrow{H}, T) - G_0^s(\overrightarrow{H}, T) = (\varepsilon_v - H\Phi_0) L_z \quad \text{--- (6.138)} \]

The Gibbs free energy increased by introducing a vortex due to the electromagnetic energy \( \varepsilon_v L_z \).
When \( G_s^1(\vec{H}, T) = G_s^0(\vec{H}, T) \), having a vortex is thermodynamically the lowest energy state and vortices will be present in the superconductor. This happens as soon as the applied field exceeds the value

\[
H_{c1} = \frac{\mathcal{E}_V}{\Phi_0}
\]  

where \( H_{c1} \) is the lower critical field.

Using the expression for the energy per unit length of the vortex given by Eq (6.74), we find

\[
\mathcal{E}_V = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} K_0\left(\frac{\xi}{\lambda}\right) \quad (6.74)
\]

\[
H_{c1} = \frac{\Phi_0}{4\pi\mu_0\lambda^2} K_0\left(\frac{\xi}{\lambda}\right) \quad (6.140)
\]

For high-\( \kappa \) (\( \lambda/\xi \gg 1 \)) superconductors

\[
\lim_{x \to 0} K_0(x) \to -\ln x
\]

\[
\lim_{\lambda \gg \xi} H_{c1} = \frac{\Phi_0}{4\pi\mu_0\lambda^2} \ln\left(\frac{\lambda}{\xi}\right) \quad (6.141)
\]
Hence for $H < H_{c1}$, no vortices enter the superconductor ($B \approx 0$ – Meissner state).

For $H \geq H_{c1}$ vortices enter and the flux density is no longer zero.

Because both $\xi$ and $\lambda$ depend on temperature, so does $H_{c1}(T)$, thereby forming the phase boundary in the $H$–$T$ plane shown in Figure 6.15.
Upper critical field ($H_{c2}$)

As the applied magnetic field increases, the average flux density increases in the superconductor because the density of the vortices, $n_v$, increase. According to Eq(6.2),

$$\mathcal{B} = n_v \Phi_v \quad - \quad (6.142)$$

where $\mathcal{B} = <B>$. 

Figure 6.16 The top view of a superconductor with the field coming out of the page. The vortices form a triangular array with the separation $\alpha$ between vortices.
Now let us try to estimate the upper critical field, $H_{c2}$. As the applied field increases, the vortices in the triangular lattice get closer together. The flux density $B$ averaged over the sample is given by Eq.(6.142)

$$B = \frac{2\Phi_0}{\sqrt{3}\alpha^2}$$  — (6.143)

As the applied field increases enough so that the cores begin to overlap, the whole superconductor begins to be covered with the normal cores and hence becomes normal. At this point $B=\mu_0H$ because the material is normal.

Specifically with $\alpha \approx 2\xi$ the cores begin to overlap so that at the upper critical field

$$H_{c2} = \frac{2\Phi_0}{\sqrt{3}\mu_0(2\xi)^2} \approx \frac{\Phi_0}{2\sqrt{3}\mu_0\xi^2}$$  — (6.144)

the material reverts to the normal state. Notice that our estimate of $H_{c2}$ depends on the model with which we describe the vortices.

From the Ginzburg–Landau calculation:

$$H_{c2} = \frac{\Phi_0}{2\pi\mu_0\xi^2}$$  — (6.145)
Temperature dependence of $H_{c2}$

Since

$$\lim_{T \to T_c} \xi(T) \sim \frac{1}{\sqrt{1 - (T/T_c)}}$$

we find that

$$\lim_{T \to T_c} H_{c2}(T) \propto \frac{1}{\xi^2} \propto 1 - \frac{T}{T_c}$$

This linear dependence on temperature near $T_c$ is also apparent in the figure.

Figure 6.15 The $H$–$T$ phase diagram for a bulk superconducting slab in a uniform magnetic field for a type II superconductor.